Infinite volume limit for the dipole gas

J. Dimock
Dept. of Mathematics
SUNY at Buffalo
Buffalo, NY 14260

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Abstract

We consider a classical lattice dipole gas with low activity and show that the pressure has a limit as the volume goes to infinity. The result is obtained by a renormalization group analysis of the model.

1 Introduction

1.1 overview

We study a dipole gas on a unit lattice $\mathbb{Z}^d$ with $d \geq 3$. The potential between unit dipoles with moments $p_1, p_2 \in \mathbb{S}^{d-1}$ at positions $x, y \in \mathbb{Z}^d$ is

$$(p_1 \cdot \partial)(p_2 \cdot \partial)C(x - y) \tag{1}$$

where $C(x - y)$ is the Coulomb potential, that is the kernel of the inverse Laplacian

$$C(x, y) = (-\Delta)^{-1}(x, y) = (2\pi)^{-d} \int_{[-\pi,\pi]^d} \frac{e^{ip(x-y)}}{2\sum_\mu (1 - \cos p_\mu)} dp \tag{2}$$

For this potential we consider the dipole gas in the grand canonical ensemble. Let $\Lambda_N \subset \mathbb{R}^d$ be a box of the form

$$\Lambda_N = \left[\frac{-L^N}{2}, \frac{L^N}{2}\right]^d \tag{3}$$

where L is large, odd, and positive. For $\Lambda_N \cap \mathbb{Z}^d$ the grand canonical partition function with activity $z > 0$ and (for convenience) inverse temperature $\beta = 1$ can be represented as a Euclidean field theory and is given by

$$Z_N = \int \exp\left(zW(\Lambda_N, \phi)\right) d\mu_C(\phi) \tag{4}$$

where

$$W(\Lambda_N, \phi) = 2 \int_{\mathbb{S}^{d-1}} dp \sum_{x \in \Lambda_N \cap \mathbb{Z}^d} \cos(p \cdot \partial\phi(x)) \tag{5}$$

Here dp is the normalized rotation invariant measure on $\mathbb{S}^{d-1}$. The fields $\phi(x)$ are a family of Gaussian random variables indexed by $x \in \mathbb{Z}^d$ with mean zero and covariance given by the positive definite function $C(x, y)$. The measure μ_C is the underlying measure. To make the connection with the dipole gas one expands the exponential in (4) and carries out the Gaussian integrals. Similarly one can define correlation functions in terms of the field theory.

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One would like to take the thermodynamic limit for these quantities, that is the limit as $N \to \infty$. Actually Z_N itself has no limit but there should be a limit for the pressure defined by

$$p_N = |\Lambda_N|^{-1} \log Z_N \tag{6}$$

as well as for the correlation functions. Such limits have been obtained by Fröhlich and Park [12] and by Fröhlich and Spencer [13] using a method of correlation inequalities.

In this paper we want to study the problem by a more robust method which is capable of answering other questions about the long distance behavior of the model such as decay of correlations. If the potential were integrable one could establish such results with a Mayer expansion. However the long distance behavior $\partial_\mu \partial_\nu C(x - y) = \mathcal{O}(|x - y|^{-d})$ is not integrable. Insead we use the method of the renormalization group (RG). The basic idea is to break up the integral into a sequence of more controllable integrals and analyze the effects separately at each stage.

We follow particularly a RG approach for low activity recently developed by Brydges and Slade [2], [3]. A collateral benefit of this paper is to work out some details of their method in case of the dipole gas. Earlier work on the RG approach to the dipole gas can be found in Gawedski and Kupiainen [14], Brydges and Yau [9], Dimock and Hurd [10], and Brydges and Keller [6] .

In all these treatments the model is either defined on the torus $\mathbb{R}^d/L^N\mathbb{Z}^d$ with a momentum cutoff or on a toroidal lattice $\mathbb{Z}^d/L^N\mathbb{Z}^d$. One obtains bounds on the partition function and correlation functions uniform in N. As explained above we work on $\mathbb{Z}^d$ with the interaction confined to a finite volume Λ_N. We essentially reproduce the basic torus results, at least for the partition function, but then also take the $N \to \infty$ limit. The $N \to \infty$ limit would be awkward for a sequence of tori because the N dependence appears in the covariance C as well as the interaction. Furthermore for the tori there are difficulties connected with the change in topology. The disadvantage for us is that our finite volume approximation loses some translation invariance because of the boundary. Since translation invariance is a key ingredient in the proof, dealing with this loss is one of the main issues.

Besides the dipole gas papers mentioned above we cite some other papers which treat infrared problems by RG techniques. There is the work of Brydges, Dimock, and Hurd [4],[5], Brydges, Mitter, and Scoppola [8], and Abdesselam [1] on non-Gaussian fixed points for ϕ^4 models, and Dimock and Hurd [11] on Sine-Gordon models in $d = 2$ (the Coulomb gas), and Mitter and Scoppola [15] on self-avoiding random walks. These papers either either work in a finite volume and get bounds uniform in the volume or else work with a formal infinite volume limit. The hope is that the techniques of the present paper point the way to carrying these results over to an actual infinite volume limit.

1.2   the main result

We now state the main result. For our renormalization group approach we use a different finite volume approximation than (4) following the analysis of Brydges [2]. We first add a term $(1 - \epsilon)V(\Lambda_N, \phi)$ where

$$V(\Lambda_N, \phi) = \frac{1}{4} \sum_{x \in \Lambda_N \cap \mathbb{Z}^d} \sum_{\pm\mu=1}^{d} (\partial_\mu \phi(x))^2 \tag{7}$$

Here $\partial_\mu \phi$ is either the forward or backward lattice derivative along the unit basis vector e_μ defined by

$$\partial_\mu \phi(x) = \phi(x + e_\mu) - \phi(x) \tag{8}$$

where $e_{-\mu} = -e_\mu$. Then ∂_μ and $\partial_{-\mu}$ are adjoint to each other and $-\Delta = 1/2 \sum_\mu \partial_\mu^* \partial_\mu$. [1]

This addition of $(1 - \epsilon)V(\Lambda_N, \phi)$ is partially compensated by replacing the covariance C by $\epsilon^{-1}C$. Thus instead of (4) we consider

$$Z_N' = \int \exp\left(zW(\Lambda_N, \phi) - (1 - \epsilon)V(\Lambda_N, \phi)\right) d\mu_{\epsilon^{-1}C}(\phi) \tag{9}$$

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[1]We distinguish forward and backward derivatives to facilitate a symmetric decomposition of $V(\Lambda_N)$ into blocks

Then divide by

$$Z_N'' = \int \exp\Big(-(1-\epsilon)V(\Lambda_N, \phi)\Big) d\mu_{\epsilon^{-1}C}(\phi) \tag{10}$$

and form a new finite volume partition function

$$Z_N = Z_N'/Z_N'' \tag{11}$$

Since formally $(Z_N'')^{-1} \exp\Big(-(1-\epsilon)V(\Lambda_N)\Big) d\mu_{\epsilon^{-1}C}$ converges to $d\mu_C$, so formally Z_N yields the same limit as (4). This holds for any choice of ϵ; the choice of ϵ is a choice of how much $(\partial\phi)^2$ one is putting in the measure and how much in the interaction.

The point of the adjustment is that one can make a shrewd choice of ϵ to facilitate the analysis. The main result is:

Theorem 1 *For $|z|$ sufficiently small there is a $\epsilon = \epsilon(z)$ close to 1 so that the pressure $p_N = |\Lambda_N|^{-1} \log Z_N$ has a limit as $N \to \infty$.*

The proof will involve a demonstration that with the proper choice of $\epsilon = \epsilon(z)$ the density $\exp\Big(zW - (1-\epsilon)V\Big)$ tends to zero under the RG flow leaving a measure like $\mu_{\epsilon(z)^{-1}C}$ to describe the long distance behavior of the system. Accordingly $\epsilon(z)$ is interpreted as a dielectric constant. To make this remark precise one would have to study the correlation functions by these methods. This seems quite feasible, but we do not develop this aspect.

For the proof of the theorem it is convenient to rewrite the partition function. We first scale $\phi \to \phi/\sqrt{\epsilon}$ and then put $\sigma = \epsilon^{-1} - 1$. Then we have

$$
\begin{aligned}
Z_N'(z, \sigma) &= \int \exp\Big(zW(\Lambda_N, \sqrt{1+\sigma}\phi) - \sigma V(\Lambda_N, \phi)\Big) d\mu_C(\phi) \\
Z_N''(\sigma) &= \int \exp\Big(-\sigma V(\Lambda_N, \phi))\Big) d\mu_C(\phi) \\
Z_N(z, \sigma) &= Z_N'(z, \sigma)/Z_N''(\sigma)
\end{aligned}
\tag{12}
$$

Then the problem is to show that for $|z|$ sufficiently small there is a (smooth) $\sigma = \sigma(z)$ near zero such that with this choice of σ

$$|\Lambda_N|^{-1} \log Z_N(z, \sigma(z)) = |\Lambda_N|^{-1} \log Z_N'(z, \sigma(z)) - |\Lambda_N|^{-1} \log Z_N''(\sigma(z)) \tag{13}$$

has a limit as $N \to \infty$. The two terms are treated separately and theorem 1 is proved by taking $\epsilon(z) = (1 + \sigma(z))^{-1}$.

The paper is organized as follows. In section 2 we show that the normalizing factor $|\Lambda_N|^{-1} \log Z_N''$ has a limit. In section 3 we give some general definitions and estimates and define the basic RG transformation. In section 4 we perform the detailed analysis of the RG transformation isolating the leading terms. In section 5 we study the flow of the renormalization group and find the stable manifold $\sigma = \sigma(z)$. Finally in section 6 we assemble the results and prove the limit for $|\Lambda_N|^{-1} \log Z_N'$.

2 The normalizing factor

We consider the infinite volume limit for the normalizing factor $|\Lambda_N|^{-1} \log Z_N''(\sigma)$. This is the problem of the infinite volume limit for a finite volume perturbation in the field strength and may be of more general interest.

First we realize the Gaussian process as given by $\phi = C^{1/2}Y$ where Y has identity covariance. Then

$$Z_N''(\sigma) = \int \exp\Big(-\frac{\sigma}{2}(Y, T_N Y)\Big) d\mu_I(Y) \tag{14}$$

where T_N is the positive operator

$$T_N = \frac{1}{2} \sum_{\pm\mu=1}^{d} C^{1/2} \partial_\mu^* 1_{\Lambda_N} \partial_\mu C^{1/2} \tag{15}$$

and 1_{Λ_N} is the characteristic function of Λ_N.

Lemma 1 *The operator T_N on $\ell^2(\mathbb{Z}^d)$ has the properties*

1. *$\mathrm{tr}\, T_N = |\Lambda_N|$*

2. *$\|T_N\| \leq 1$.*

Proof. T_N is trace class since 1_{Λ_N} is trace class and $\partial_\mu C^{1/2}$ is bounded. Since $[\partial_\mu, C] = 0$ we have

$$\mathrm{tr}\, T_N = \frac{1}{2} \sum_{\pm\mu=1}^{d} \mathrm{tr}\, \left(\partial_\mu^* \partial_\mu C 1_{\Lambda_N} \right) = \mathrm{tr}\, 1_{\Lambda_N} = |\Lambda_N| \tag{16}$$

The bound $\|T_N\| \leq 1$ follows from

$$\begin{aligned}
|(h, T_N f)| &\leq \frac{1}{2} \sum_\mu |(\partial_\mu C^{1/2} h)(x)|^2 \chi_\Lambda(x) |(\partial_\mu C^{1/2} f)(x)|^2 \\
&\leq \left(\frac{1}{2} \sum_\mu \|\partial_\mu C^{1/2} h\|^2 \right)^{1/2} \left(\frac{1}{2} \sum_\mu \|\partial_\mu C^{1/2} f\|^2 \right)^{1/2} \\
&= \|h\| \|f\|
\end{aligned} \tag{17}$$

Theorem 2 *For real σ with $|\sigma| < 1$, $|\Lambda_N|^{-1} \log Z_N''(\sigma)$ converges as $N \to \infty$.*

Proof. Since T_N is trace class and

$$\|f\|^2 - \sigma(f, T_N f) \geq (1 - |\sigma|) \|f\|^2 > 0 \tag{18}$$

the integral defining $Z_N''(\sigma)$ in (14) exists and can be evaluated as

$$Z_N''(\sigma) = \det(1 + \sigma T_N)^{-1/2} \tag{19}$$

(See for example [17]). Furthermore since $|\sigma| \|T_N\| \leq |\sigma| < 1$ we have the expansion

$$Z_N''(\sigma) = \exp\left(\frac{1}{2} \sum_{n=1}^\infty \frac{(-\sigma)^n}{n} \mathrm{tr}\, (T_N^n) \right) \tag{20}$$

(See for example [16]). Hence

$$|\Lambda_N|^{-1} \log Z_N''(\sigma) = \frac{1}{2} \sum_{n=1}^\infty \frac{(-\sigma)^n}{n} \frac{\mathrm{tr}\, (T_N^n)}{|\Lambda_N|} \tag{21}$$

We have with the trace norm $\|\cdot\|_1$

$$|\mathrm{tr}\, (T_N^n)| \leq \|T_N^n\|_1 \leq \|T_N\|_1 \|T_N^{n-1}\| \leq \|T_N\|_1 \leq |\Lambda_N| \tag{22}$$

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Hence the sum is dominated by $\sum_n |\sigma|^n < \infty$. We show below that for each $n \geq 1$

$$a_n = \lim_{N \to \infty} \frac{\text{tr } (T_N^n)}{|\Lambda_N|} \tag{23}$$

exists. Then by the dominated convergence theorem we have the existence of

$$\lim_{N \to \infty} |\Lambda_N|^{-1} \log Z_N''(\sigma) = \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-\sigma)^n}{n} a_n \tag{24}$$

Now consider the convergence (23). We write

$$\text{tr } (T_N^n) = 2^{-n} \sum_{\mu_1, \dots, \mu_n} \text{tr } (1_{\Lambda_N} \Pi_{\mu_1 \mu_2} \cdots 1_{\Lambda_N} \Pi_{\mu_n \mu_1}) \tag{25}$$

where the sums are over $\pm \mu = 1, \dots, d$ and

$$\Pi_{\mu\nu} = \partial_\mu C \partial_\nu^* \tag{26}$$

We rewrite this as

$$\text{tr } (T_N^n) = \sum_{x \in \Lambda_N} a_n^N(x) \tag{27}$$

where

$$a_n^N(x_1) = 2^{-n} \sum_{\mu_1, \dots, \mu_n} \sum_{x_2, \dots, x_n \in \Lambda_N} \Pi_{\mu_1 \mu_2}(x_1 - x_2) \Pi_{\mu_2 \mu_3}(x_2 - x_3) \cdots \Pi_{\mu_n \mu_1}(x_n - x_1) \tag{28}$$

The quantity a_n is the same expression without the restriction to Λ_N. It is independent of x_1 and we can take $x_1 = 0$. Thus it is

$$a_n = 2^{-n} \sum_{\mu_1, \dots, \mu_n} \sum_{x_2, \dots, x_n} \Pi_{\mu\mu_2}(-x_2) \Pi_{\mu_2 \mu_3}(x_2 - x_3) \cdots \Pi_{\mu_n \mu}(x_n) \tag{29}$$

To see that a_n is finite we use (see lemma 2 to follow)

$$|\Pi_{\mu\nu}(x - y)| \leq C(1 + |x - y|)^{-d} \tag{30}$$

then in (29) we use the estimate [2]

$$\sum_y (1 + |x - y|)^{-d})(1 + |y|)^{-d+k\delta} \leq C_{k,\delta}(1 + |x|)^{-d+(k+1)\delta} \tag{31}$$

valid for $k\delta < d$. Applying this successively to $x_n, x_{n-1}, \dots$ we are left with

$$\int (1 + |x_2|)^{-2d+(n-1)\delta} dx_2 \tag{32}$$

which is finite if $(n-1)\delta < d$. Thus a_n is finite. Similarly one shows that $|a_n^N(x)|$ is bounded uniformly in N.

Now we write

$$|\Lambda_N|^{-1} \text{tr } (T_N^n) = a_n + |\Lambda_N|^{-1} \sum_{x_1 \in \Lambda_N} (a_n^N(x_1) - a_n) \tag{33}$$

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[2] To prove it divide the summation region into $|y| \leq |x|/2$ and the complement

We show that the second term above goes to zero as $N \to \infty$ to complete the proof.

First define a slightly smaller volume

$$\Lambda_N^* = \left[-\frac{L^N}{2} + N, \frac{L^N}{2} - N \right]^d \tag{34}$$

The contribution from $x_1 \notin \Lambda_N^*$ is bounded by

$$|\Lambda_N|^{-1} \sum_{x_1 \in \Lambda_N - \Lambda_N^*} |a_n^N(x_1) - a| \leq \mathcal{O}(1) \frac{|\Lambda_N - \Lambda_N^*|}{|\Lambda_N|} \leq \mathcal{O}(NL^{-N}) \tag{35}$$

which goes to zero.

Now suppose that $x_1 \in \Lambda_N^*$. Then we have

$$a_n^N(x_1) - a_n = 2^{-n} \sum_{\mu_1, \dots, \mu_n} \sum_{(x_2, \dots, x_n) \in ((\Lambda_N)^{n-1})^c} \Pi_{\mu_1 \mu_2}(x_1 - x_2) \Pi_{\mu_2 \mu_3}(x_2 - x_3) \cdots \Pi_{\mu_n \mu_1}(x_n - x_1) \tag{36}$$

At least one variable must be in Λ_N^c, say x_k. Furthermore at least one pair of adjacent variables must satisfy $|x_j - x_{j+1}| \geq N/n$. Otherwise $|x_1 - x_k| \leq N(k-1)/n < N$ which contradicts that $x_1 \in \Lambda_N^*, x_k \in \Lambda_N^c$. Thus we can make the estimate

$$|\Pi_{\mu_j \mu_{j+1}}(x_j - x_{j+1})| \leq C(1 + |x_j - x_{j+1}|)^{-d} \leq C(1 + N/n)^{-\epsilon}(1 + |x_j - x_{j+1}|)^{-d+\epsilon} \tag{37}$$

If ϵ is small enough the the reduced decay does not affect convergence in (36). Thus we have

$$|a_n^N(x_1) - a_n| \leq \mathcal{O}(N^{-\epsilon}) \tag{38}$$

Therefore

$$|\Lambda_N|^{-1} \sum_{x_1 \in \Lambda_N^*} |a_n^N(x_1) - a| \leq \mathcal{O}(N^{-\epsilon}) \frac{|\Lambda_N^*|}{|\Lambda_N|} \leq \mathcal{O}(N^{-\epsilon}) \tag{39}$$

which also goes to zero to complete the proof.

3 Preliminaries

3.1 multiscale decomposition

Renormalization group methods are based on a multiscale decomposition of the basic lattice covariance. We choose a decomposition into finite range covariances developed by Brydges, Guadagni, and Mitter [7]. This is an alternative to block spin averaging and has the advantage of making fluctuation integrals simpler and the fluctuation covariances smoother. The smoothness is essential for the method.

The decomposition has the form

$$C(x - y) = \sum_{j=1}^{\infty} \Gamma_j(x - y) \tag{40}$$

where $\Gamma_j(x)$ is defined on $\mathbb{Z}^d$, is positive semi-definite, and satisfies $\Gamma_j(x) = 0$ if $|x| \geq L^j/2$ for some odd integer $L \geq 3$. Furthermore there is a constant c_0 independent of L such that

$$|\Gamma_j(x)| \leq c_0 L^{-(d-2)(j-1)} \tag{41}$$

for all j, x. It follows that the series converges uniformly. Let $\partial^\alpha = \prod_{\pm\mu=1}^{d} \partial_\mu^{\alpha_\mu}$ be a multi-derivative and let $|\alpha| = \sum_\mu |\alpha_\mu|$. Then there are constants c_α independent of L such that

$$|\partial^\alpha \Gamma_j(x)| \leq c_\alpha L^{-(d-2+|\alpha|)(j-1)} \tag{42}$$

Then the differentiated series converges uniformly to $\partial^\alpha C$.

An elementary consequence of this expansion is an estimate on the decay of $C(x-y)$ as $|x-y| \to \infty$:

Lemma 2 *There are constants $C_{L,\alpha}$ such that*

$$|\partial^\alpha C(x)| \leq C_{L,\alpha}(1+|x|)^{-d+2-|\alpha|} \tag{43}$$

Proof. First consider the case with no derivatives. For $|x| \geq L/2$ choose $k \geq 1$ so that $L^k/2 \leq |x| \leq L^{k+1}/2$. If $j \leq k$ then $\Gamma_j(x) = 0$ and we have

$$C(x) = \sum_{j=k+1}^{\infty} \Gamma_j(x) \tag{44}$$

This is estimated by

$$\sum_{j=k+1}^{\infty} c_0 L^{-(d-2)(j-1)} \leq 2c_0 L^{-(d-2)k} \leq c_0 L|x|^{-(d-2)} \tag{45}$$

which suffices. With derivatives we get the improved decay from (42). This completes the proof.

For the renormalization group we break off pieces of $C(x-y)$ one at a time. Accordingly we define

$$C_k(x-y) = \sum_{j=k+1}^{\infty} \Gamma_j(x-y) \tag{46}$$

Then $C = C_0$ and

$$C_k(x-y) = C_{k+1}(x-y) + \Gamma_{k+1}(x-y) \tag{47}$$

3.2   RG transformation

The partition function (12) can be written

$$Z_N'(z,\sigma) = \int \mathcal{Z}_0^N(\phi) d\mu_{C_0}(\phi) \tag{48}$$

where

$$\mathcal{Z}_0^N(\phi) = \exp\left(zW(\Lambda_N, \sqrt{1+\sigma}\phi) - \sigma V(\Lambda_N, \phi)\right) \tag{49}$$

The identity $C_0 = C_1 + \Gamma_1$ lets us replace an integral over μ_{C_0} by an integral over μ_{Γ_1} and μ_{C_1} We have

$$Z_N'(z,\sigma) = \int \mathcal{Z}_0^N(\phi + \zeta) d\mu_{\Gamma_1}(\zeta) d\mu_{C_1}(\phi)$$
$$= \int \mathcal{Z}_1^N(\phi) d\mu_{C_1}(\phi) \tag{50}$$

We have defined a new density by the fluctuation integral

$$\mathcal{Z}_1^N(\phi) = (\mu_{\Gamma_1} * \mathcal{Z}_0^N)(\phi) \equiv \int \mathcal{Z}_0^N(\phi + \zeta) d\mu_{\Gamma_1}(\zeta) \tag{51}$$

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Since Γ_1, C_1 are only positive semi-definite these are degenerate Gaussian measures. Nevertheless these integrals are well-defined and the above manipulations are valid. We discuss these issues in appendix A

Continuing in this fashion we have the representation for $j = 0, 1, 2, \ldots$

$$Z'_N(z, \sigma) = \int \mathcal{Z}_j^N(\phi) d\mu_{C_j}(\phi) \tag{52}$$

where the density $\mathcal{Z}_j^N(\phi)$ is defined by

$$\mathcal{Z}_{j+1}^N(\phi) = (\mu_{\Gamma_{j+1}} * \mathcal{Z}_j^N)(\phi) = \int \mathcal{Z}_j^N(\phi + \zeta) d\mu_{\Gamma_{j+1}}(\zeta) \tag{53}$$

Our problem is to study the growth of these densities as $j \to \infty$.

Note that we have refrained from scaling after each fluctuation integral which is the usual procedure in the renormalization group. Thus the volume stays constant but correlations weaken as we proceed.

3.3 local expansion

Each density $\mathcal{Z}_j^N(\phi)$ will be written in a form which exhibits its locality properties known as a polymer representation. The localization becomes coarser as j gets larger.

For $j = 0, 1, 2, \ldots$ we partition $\mathbb{Z}^d$ into j-blocks B. These have side L^j and are translates of

$$B_0 = \{x \in \mathbb{Z}^d : |x| < 1/2(L^j - 1)\} \tag{54}$$

by points in the lattice $L^j \mathbb{Z}^d$. The set of all j-blocks in $\Lambda = \Lambda_N$ is denoted $\mathcal{B}_j(\Lambda)$ or just $\mathcal{B}_j$. A union of j-blocks X is called a j-polymer. In particular Λ is a j-polymer for $j \leq N$. The set of all j-polymers in Λ is denoted $\mathcal{P}_j(\Lambda)$ or just $\mathcal{P}_j$. The connected j-polymers are denoted $\mathcal{P}_{j,c}$.

The number of j-blocks in a j-polymer X is denoted $|X|_j$. The j-polymer X is a small set if it is connected and $|X|_j \leq 2^d$. The set of all small set polymers is denoted $\mathcal{S}_j(\Lambda)$ or just $\mathcal{S}_j$. A j-block B has a small set neighborhood

$$B^* = \cup\{Y \in \mathcal{S}_j : Y \supset B\} \tag{55}$$

Similarly a j-polymer X has a small set neighborhood X^*.

The density $\mathcal{Z}_j^N(\phi)$ for $\phi : \mathbb{Z}^d \to \mathbb{R}$ will be written in the the general form

$$\mathcal{Z} = (I \circ K)(\Lambda) \equiv \sum_{X \in \mathcal{P}_j(\Lambda)} I(\Lambda - X)K(X) \tag{56}$$

The $I(Y)$ is a background functional which is explicitly known and carries the main contribution to the density. The $K(X)$ is called a polymer activity and represents small corrections to this background.

We assume $I(Y)$ has the form

$$I(Y) = \prod_{B \in \mathcal{B}_j : B \subset Y} I(B) \tag{57}$$

and that $I(B, \phi)$ depends on ϕ only B^*. We also assume $K(X)$ factors over the connected components $\mathcal{C}(X)$ of X, that is

$$K(X) = \prod_{Y \subset \mathcal{C}(X)} K(Y) \tag{58}$$

and that $K(X, \phi)$ only depends on ϕ in X^*.

All this is quite general. Special to our model is the fact that the background $I(B)$ has the form $I(E, \sigma, B) = \exp(-V(E, \sigma, B))$ where [3]

$$V(E, \sigma, B, \phi) = E(B) + \frac{1}{4} \sum_{x \in B} \sum_{\mu\nu} \sigma_{\mu\nu}(B)\partial_\mu\phi(x)\partial_\nu\phi(x) \tag{59}$$

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[3] Sums over μ are understood to range over $\pm\mu = 1, \ldots, d$, unless otherwise specified

for some functions $E, \sigma_{\mu\nu} : \mathcal{B}_j \to \mathbb{R}$. In fact we will usually be able to take $\sigma_{\mu\nu}(B) = \sigma\delta_{\mu\nu}$ for some constant σ in which case

$$V(E, \sigma, B, \phi) = E(B) + \frac{\sigma}{4}\sum_{x \in B}\sum_{\mu}(\partial_\mu\phi(x))^2 \equiv E(B) + \sigma V(B) \tag{60}$$

Also in our model we will have

$$\begin{aligned} K(X, \phi) &= K(X, -\phi) \\ K(X, \phi) &= K(X, \phi + c) \end{aligned} \tag{61}$$

The second holds for any constant c and is equivalent to saying that $K(X, \phi)$ only depends on derivatives $\partial\phi$.

3.4 norms

We define a menagerie of norms following Brydges [2].

3.4.1

If X is a j-polymer we consider the Banach space $\Phi_j(X)$ of functions $\phi : X \to \mathbb{R}$ modulo constants with the norm

$$\|\phi\|_{\Phi_j(X)} = h_j^{-1}\max\left\{\|\nabla_j\phi\|_{X,\infty}, \ \|\nabla_j^2\phi\|_{X,\infty}\right\} \tag{62}$$

where

$$\begin{aligned} \|\nabla_j\phi\|_{X,\infty} &= \sup_{x \in X, \mu}|\nabla_{j,\mu}\phi(x)| \\ \nabla_{j,\mu} &= L^j\partial_\mu \qquad h_j = L^{-(d-2)j/2}h \end{aligned} \tag{63}$$

Note that if X is also a $j+1$ polymer then we can consider $\|\phi\|_{\Phi_{j+1}(X)}$. Since $h_j^{-1} = L^{-(d-2)/2}h_{j+1}^{-1}$ and $\nabla_j = L^{-1}\nabla_{j+1}$ we have the contractive property

$$\|\phi\|_{\Phi_j(X)} \leq L^{-d/2}\|\phi\|_{\Phi_{j+1}(X)} \tag{64}$$

3.4.2

Now consider polymer activities $K(X, \phi)$ for $X \in \mathcal{P}_j$. We assume that $K(X, \phi)$ only depends on ϕ in X^* and is a $\mathcal{C}^3$ function on $\Phi_j(X^*)$.

1. For $n = 0, 1, 2, 3$ let $K_n(X, \phi)$ be the nth derivative with respect to ϕ. It is a multi-linear functional on $f_i \in \Phi_j(X^*)$ given by

$$K_n(X, \phi; f_1, \dots, f_n) = \frac{\partial^n}{\partial t_1 \dots \partial t_n}K(X, \phi + t_1 f_1 + \dots + t_n f_n)\Big|_{t_i=0} \tag{65}$$

   We define

$$\|K_n(X, \phi)\|_j = \sup\{ |K_n(X, \phi; f_1, \dots f_n)| : \|f_j\|_{\Phi_j(X^*)} \leq 1\} \tag{66}$$

2. Next define

$$\|K(X, \phi)\|_j = \sum_{n=0}^{3}\frac{1}{n!}\|K_n(X, \phi)\|_j \tag{67}$$

   This combination of derivatives has the multiplicative property

$$\|K(X, \phi)H(Y, \phi)\|_j \leq \|K(X, \phi)\|_j\|H(Y, \phi)\|_j \tag{68}$$

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3. Next we pick a large field regulator $G_j(X, \phi', \zeta)$ which depends on ϕ', ζ in X^*. It is assumed to have the form $G_j(X, \phi', \zeta) = G_j(X, \phi', 0)G_j(X, 0, \zeta)$ and satisfy $G_j(X, \phi', \zeta) \geq 1$ and $G_j(X, 0, 0) = 1$. A polymer activity $K(X, \phi)$ is regarded as a function $K(X, \phi' + \zeta)$ of ϕ', ζ and we define a norm

$$\|K(X)\|_j = \sup_{\phi', \zeta} \|K_n(X, \phi' + \zeta)\|_j G_j(X, \phi', \zeta)^{-1} \tag{69}$$

Sometimes we want to consider the same norm but with the polymer activity as a function of ϕ' only. In this case we put a prime on the norm and define

$$\begin{aligned} \|K(X)\|'_j &= \sup_{\phi', \zeta} \|K_n(X, \phi')\|_j G_j(X, \phi', \zeta)^{-1} \\ &= \sup_{\phi'} \|K_n(X, \phi')\|_j G_j(X, \phi', 0)^{-1} \end{aligned} \tag{70}$$

For large field regulators there are two choices. The strong regulator is

$$G_{s,j}(X, \phi', \zeta) = \prod_{B \in \mathcal{B}_j(X)} \exp\left(\|\phi'\|^2_{\Phi_j(B^*)} + \|\zeta\|^2_{\Phi_j(B^*)}\right) \tag{71}$$

The weak regulator is

$$\begin{aligned} G_j(X, \phi', \zeta) = &\prod_{B \in \mathcal{B}_j(X)} \exp\left(c_1 h_j^{-2} L^{-dj} \|\nabla_j \phi'\|^2_{B,2} + c_2 h_j^{-2} \|\nabla_j^2 \phi'\|^2_{B^*,\infty}\right) \\ &\times \exp\left(c_3 h_j^{-2} L^{-(d-1)j} \|\nabla_j \phi'\|^2_{\partial X, 2}\right) \\ &\times \prod_{B \in \mathcal{B}_j(X)} \exp\left(c_4 h_j^{-2} \max_{0 \leq p \leq 2} \|\nabla_j^p \zeta\|^2_{B^*,\infty}\right) \end{aligned} \tag{72}$$

(Note that $h_j^{-2} L^{-dj} \|\nabla_j \phi'\|^2_{B,2} = h^{-2} \|\partial \phi'\|^2_{B,2}$ actually has no explicit j-dependence. Nevertheless it is convenient to write it in this fashion.) The norm with strong regulator is denoted $\|K(X)\|_{s,j}$, and the norm with the weak regulator is denoted just $\|K(X)\|_j$. We note also ([2], (6.100)) that

$$G_{s,j}(X) \leq G_{s,j}(X)^2 \leq G_j(X) \tag{73}$$

and hence

$$\|K(X)\|_j \leq \|K(X)\|_{s,j} \tag{74}$$

4. Finally for the weak norm we define for $A \geq 1$

$$\|K\|_j = \sup_{X \in \mathcal{P}_{j,c}} \|K(X)\|_j A^{|X|_j} \tag{75}$$

where the supremum is over connected j-polymers X. Polymer activities $K(X, \phi)$ defined on connected j-polymers $X \subset \Lambda_N$ with this norm constitute a Banach space denoted $\mathcal{K}_j(\Lambda_N)$.

3.4.3

The norms are defined to satisfy the following properties which hold for suitable choices of c_1, c_2, c_3, c_4, L sufficiently large, and h sufficiently large depending on L. For the proofs see [2].

- If $\mathcal{C}(X)$ are the connected components of X then

$$\|K(X)\|_j \leq \prod_{Y \in \mathcal{C}(X)} \|K(Y)\|_j \tag{76}$$

- If X, Y are disjoint (but possibly touching)

$$\left\| \left(\prod_{B \subset X} F(B) \right) K(Y) \right\|_j \le \prod_{B \subset X} \|F(B)\|_{s,j} \|K(Y)\|_j \tag{77}$$

- If

$$K^{\#}(X, \phi) = \int K(X, \phi, \zeta) d\mu_{\Gamma_{j+1}}(\zeta) \tag{78}$$

then

$$\|K^{\#}(X)\|_j' \le 2^{|X|_j} \|K(X)\|_j \le (A/2)^{-|X|_j} \|K\|_j \tag{79}$$

- Suppose that U is a $(j+1)$-polymer and hence a j-polymer. Then

$$\|K(U)\|_{j+1} \le \|K(U)\|_j' \tag{80}$$

also for the strong norm.

3.5 estimates

We illustrate the use of these norms with some estimates we will need. We work in somewhat more generality than we need by introducing potentials of the form

$$V(s, B, \phi) = \frac{1}{4} \sum_{x \in B} \sum_{\mu\nu} s_{\mu\nu}(x) \partial_\mu \phi(x) \partial_\nu \phi(x) \tag{81}$$

The functions $s_{\mu\nu}(x)$ are normed by

$$\|s\|_j = \sup_{B \in \mathcal{B}_j} |B|^{-1} \|s\|_{1,B} = \sup_{B \in \mathcal{B}_j} L^{-dj} \sum_{\mu\nu} \sum_{x \in B} |s_{\mu\nu}(x)| \tag{82}$$

Note that if $s_{\mu\nu}(x) = \sigma \delta_{\mu\nu}$ then $V(s, B) = \sigma V(B)$ as defined in (60) and the norm is $\|s\|_j = 2d\,\sigma$.

Lemma 3

1. *For any $s_{\mu\nu}(x)$*
$$\|V(s, B)\|_{s,j}' \le h^2 \|s\|_j \qquad \|V(s, B)\|_{s,j} \le h^2 \|s\|_j \tag{83}$$

2. *The function $\sigma \to \exp(-\sigma V(B))$ is complex analytic and if $h^2 \sigma$ is sufficiently small*
$$\|e^{-\sigma V(B)}\|_{s,j}' \le 2 \qquad \|e^{-\sigma V(B)}\|_{s,j} \le 2 \tag{84}$$

Proof. Start with the estimate for $x \in B$

$$|\partial_\mu \phi(x)| = L^{-j} |\nabla_{j,\mu} \phi(x)| \le h_j L^{-j} \|\phi\|_{\Phi_j(B^*)} = h L^{-dj/2} \|\phi\|_{\Phi_j(B^*)} \tag{85}$$

The first derivative is $[\partial_\mu \phi(x)]_1(f) = \partial_\mu f(x)$ and it satisfies $|[\partial_\mu \phi(x)]_1(f)| \le h L^{-dj/2} \|f\|_{\Phi_j(B^*)}$. Hence

$$\|[\partial_\mu \phi(x)]_1\|_j \le h L^{-dj/2} \tag{86}$$

Adding the derivatives

$$\|\partial_\mu \phi(x)\|_j \le h L^{-dj/2} \left(1 + \|\phi\|_{\Phi_j(B^*)} \right) \tag{87}$$

Now we estimate

$$
\begin{aligned}
\|V(s,B,\phi)\|_j \leq & \frac{1}{4}\sum_{\mu\nu}\sum_{x\in B}|s_{\mu\nu}(x)|h^2 L^{-dj}\Big(1+\|\phi\|_{\Phi_j(B^*)}\Big)^2 \\
\leq & \frac{1}{2}h^2\|s\|_j\Big(1+\|\phi\|^2_{\Phi_j(B^*)}\Big) \\
\leq & \frac{1}{2}h^2\|s\|_j G_{s,j}(B,\phi,0)
\end{aligned}
\tag{88}
$$

which gives $\|V(s,B)\|'_{s,j} \leq \frac{1}{2}h^2\|s\|_j$. Similarly

$$
\begin{aligned}
\|V(s,B,\phi'+\zeta)\|_j \leq & \frac{1}{2}h^2\|s\|_j(1+\|\phi'+\zeta\|^2_{\Phi_j(B^*)}) \\
\leq & h^2\|s\|_j(1+\|\phi'\|^2_{\Phi_j(B^*)}+\|\zeta\|^2_{\Phi_j(B^*)}) \\
\leq & h^2\|s\|_j G_{s,j}(B,\phi',\zeta)
\end{aligned}
\tag{89}
$$

which gives $\|V(s,B)\|_{s,j} \leq h^2\|s\|_j$.

For the exponential estimates one can compute the derivatives, estimate, and resum (see [4] for details). Using also (88) yields

$$
\begin{aligned}
\frac{3^n}{n!}\|(e^{-\sigma V(B,\phi)})_n\|_j \leq & \exp\left(\sum_n \frac{3^n}{n!}|\sigma|\|V_n(B,\phi)\|_j\right) \\
\leq & \exp\left(9|\sigma|\|V(B,\phi)\|_j\right) \\
\leq & \exp\left(9dh^2|\sigma|(1+\|\phi\|^2_{\Phi_j(B^*)})\right) \\
= & \exp\left(9dh^2|\sigma|\right)\ G_{s,j}(B,\phi,0)
\end{aligned}
\tag{90}
$$

Now multiply by 3^{-n} and sum over n to obtain for $3/2\exp(9dh^2|\sigma|) \leq 2$

$$
\|e^{-\sigma V(B,\phi)}\|_{s,j} \leq 2\ G_{s,j}(B,\phi,0)
\tag{91}
$$

which implies $\|e^{-\sigma V(B)}\|'_{s,j} \leq 2$. The bound $\|e^{-\sigma V(B)}\|_{s,j} \leq 2$ follows similarly. This completes the proof.

We also need an estimate on the initial interaction. In this case $B \in \mathcal{B}_0$ is single site x and we consider

$$
W(u,B,\phi) = 2\int_{\mathbb{S}^{d-1}} dp\cos(p\cdot\partial\phi(x)u)
\tag{92}
$$

Lemma 4

1. $W(u,B)$ satisfies

$$
\|W(u,B)\|_{s,0} \leq 2e^{\sqrt{d}hu}
\tag{93}
$$

   $W(u,B)$ is strongly continuously differentiable in u.

2. $e^{zW(u,B)}$ is complex analytic in z and satisfies for $|z|$ is sufficiently small (depending on d,h,u)

$$
\|e^{zW(u,B)}\|_{s,0} \leq 2
\tag{94}
$$

   $e^{zW(u,B)}$ is strongly continuously differentiable in u

Proof. (1.) A calculation using $\sum_\mu |p_\mu| \le \sqrt{d}$ gives $\|[\cos(p \cdot \partial\phi(x)u)]_n\|_0 \le (\sqrt{d}hu)^n$ and so

$$\|W(u, B, \phi)\|_0 \le 2 \sup_p \| \cos(p \cdot \partial\phi(x)u)\|_0 \le 2e^{\sqrt{d}hu} \tag{95}$$

This gives the required $\|W(u,B)\|_{s,0} \le 2e^{\sqrt{d}hu}$.

We first compute the pointwise derivative in u which is

$$W'(u, B, \phi) = -2 \int_{\mathbb{S}^{d-1}} dp \sin\left(p \cdot \partial\phi(x)u\right)\left(p \cdot \partial\phi(x)\right) \tag{96}$$

Then by (87) at $j = 0$ and (95) with sine instead of cosine

$$\|W'(u, B, \phi)\|_0 \le 2e^{\sqrt{d}hu}h\left(1 + \|\phi\|_{\Phi_0(B^*)}\right) \tag{97}$$

and hence

$$\|W'(u, B)\|_{s,0} \le 4he^{\sqrt{d}hu} \tag{98}$$

Higher derivatives are treated similarly. In particular for the second derivative

$$\|W''(u, B)\|_{s,0} \le 8h^2 e^{\sqrt{d}hu} \tag{99}$$

To see that the pointwise derivative is also the strong derivative we write

$$W(u + \delta, B) - W(u, B) - \delta\, W'(u, B) = \int_0^\delta dt \int_u^{u+t} W''(s, B)ds \tag{100}$$

Inserting the bound on W'' the norm of the expression is $\mathcal{O}(\delta^2)$ which gives the result. The strong continuity of W' also follows from the bound on W''.

(2.) For the exponential bound instead of the norm $\|\cdot\|_0$ with G_0 it suffices to use the $G = 1$ norm

$$\|W(B)\|_{00} = \sup_\phi \|W(B, \phi)\|_0 = \sup_{\phi',\zeta} \|W(B, \phi' + \zeta)\|_0 \tag{101}$$

This is a stronger norm in the sense that $\|W(B)\|_{s,0} \le \|W(B)\|_{00}$. We still have $\|W(u, B)\|_{00} \le 2e^{\sqrt{d}hu}$ from (95). The new norm is multiplicative and so

$$\|e^{zW(u,B)}\|_{00} \le \sum_{n=0}^\infty \frac{|z|^n}{n!}\|W(u, B)\|_{00}^n \le \sum_{n=0}^\infty \frac{(2|z|e^{\sqrt{d}hu})^n}{n!} = \exp\left(2|z|e^{\sqrt{d}hu}\right) \tag{102}$$

This implies the same result for $\|e^{zW(u,B)}\|_{s,0}$.

The pointwise derivative in u is $(e^{zW(u)})' = zW'(u)e^{zW(u)}$ and so

$$\|(e^{zW(u,B)})'\|_{s,0} \le |z|\|W'(u, B)\|_{s,0}\|e^{zW(u,B)}\|_{00} \tag{103}$$

which we bound by (98) and (102). There is a similar bound on the second derivative which we use as before to show that the pointwise derivative is a strong derivative.

4 Analysis of the RG transformation

4.1

We now explain the Brydges-Slade RG analysis, at the same time noting the modifications due to boundary effects.

Suppose we have $\mathcal{Z}(\phi) = (I \circ K)(\Lambda, \phi)$ with polymers on scale j. This is transformed to

$$\mathcal{Z}'(\phi') = (\mu_{\Gamma_{j+1}} * \mathcal{Z})(\phi') \equiv \int \mathcal{Z}(\phi' + \zeta) d\mu_{\Gamma_{j+1}}(\zeta) \tag{104}$$

which we seek to write in the form $\mathcal{Z}'(\phi') = (I' \circ K')(\Lambda, \phi')$ where the polymers are now on scale $j+1$.

Further suppose we have picked I' and we seek K' so the identity holds. Our choice of I' is taken to have the form

$$I'(B', \phi') = \prod_{B \in \mathcal{B}_j, B \subset B'} \tilde{I}(B, \phi') \qquad B' \in \mathcal{B}_{j+1} \tag{105}$$

We define

$$\delta I(B, \phi', \zeta) = I(B, \phi' + \zeta) - \tilde{I}(B, \phi') \tag{106}$$

We also define $\tilde{K} = K \circ \delta I$, more precisely

$$\tilde{K}(X, \phi', \zeta) = \sum_{Y \subset X} K(Y, \phi' + \zeta) \delta I^{X-Y}(\phi', \zeta) \tag{107}$$

For connected X we write

$$\tilde{K}(X, \phi', \zeta) = \sum_{B \subset X} J(B, X, \phi') + \check{K}(X, \phi', \zeta) \tag{108}$$

The quantities $J(B, X)$ will eventually be chosen to depend on K and to isolate the most important part of K for cancellation. For now $J(B, X)$ are free but we require $J(B, X) = 0$ unless $X \in \mathcal{S}_j, B \subset X$ and that $J(B, X, \phi')$ depend on ϕ' only in B^*. Given K and J the equation (108) defines $\check{K}(X)$ for X connected and for any $X \in \mathcal{P}_j$ we define

$$\check{K}(X, \phi', \zeta) = \prod_{Y \in \mathcal{C}(X)} \check{K}(Y, \phi', \zeta) \tag{109}$$

Then after using the finite range property and making some rearrangements the representation $\mathcal{Z}'(\phi) = (I' \circ K')(\Lambda, \phi)$ holds with (Brydges [2], Proposition 5.1)

$$K'(U, \phi') = \sum_{X, \chi \to U} J^{\chi}(\phi') \tilde{I}^{U-(X_\chi \cup X)}(\phi') \check{K}^{\#}(X, \phi') \qquad U \in \mathcal{P}_{j+1} \tag{110}$$

Here $\chi = (B_1, X_1, \ldots B_n, X_n)$ and the condition $X, \chi \to U$ is that $X_1, \ldots X_n, X$ be strictly disjoint and satisfy $\overline{(B_1^* \cup \cdots \cup B_n^* \cup X)} = U$. Furthermore

$$J^{\chi}(\phi') = \prod_{i=1}^{n} J(B_i, X_i, \phi')$$

$$\tilde{I}^{U-(X_\chi \cup X)}(\phi') = \prod_{B \in U-(X_\chi \cup X)} \tilde{I}(B, \phi') \tag{111}$$

where $X_\chi = \cup_i X_i$. Finally $\check{K}^{\#}(X, \phi')$ is $\check{K}(X, \phi', \zeta)$ integrated over ζ as in (78).

At this point we have K' as a function of $I, \tilde{I}, J, K$. It vanishes at the point $(I, \tilde{I}, J, K) = (1, 1, 0, 0)$ since for $U \neq \emptyset$ we cannot have both $\chi = \emptyset$ and $X = \emptyset$. We are interested in the behavior in a neighborhood of this point. We have the norm (75) on K and we also define

$$\|I\|_{s,j} = \sup_{B \in \mathcal{B}_j} \|I(B)\|_{s,j}$$

$$\|\tilde{I}\|'_{s,j} = \sup_{B \in \mathcal{B}_j} \|\tilde{I}(B)\|'_{s,j} \tag{112}$$

$$\|J\|'_j = \sup_{X \in \mathcal{S}_j, B \subset X} \|J(X, B)\|'_j$$

Then we have the following uniform smoothness result.

Theorem 3 *Let A be sufficiently large.*

1. *For $R > 0$ there is a $r > 0$ such that the following holds for all j. If $\|I - 1\|_{s,j} < r$, $\|\tilde{I} - 1\|'_{s,j} < r$, $\|J\|'_j < r$ and $\|K\|_j < r$ then $\|K'\|_{j+1} < R$. Furthermore K' is a smooth function of $I, \tilde{I}, J, K$ on this domain with derivatives bounded uniformly in j.*

2. *If also*

$$\sum_{X \in \mathcal{S}_j : X \supset B} J(B, X) = 0 \tag{113}$$

   then the linearization of $K' = K'(I, \tilde{I}, J, K)$ at $(I, \tilde{I}, J, K) = (1, 1, 0, 0)$ is

$$\sum_{X \in \mathcal{P}_{j,c}, \overline{X} = U} \left(K^\#(X) + (I^\#(X) - 1)1_{X \in \mathcal{B}_j} - (\tilde{I}(X) - 1)1_{X \in \mathcal{B}_j} - \sum_{B \subset X} J(B, X) \right) \tag{114}$$

   where

$$K^\#(X, \phi) = \int K(X, \phi + \zeta) d\mu_{\Gamma_{j+1}}(\zeta) \tag{115}$$

Proof. Brydges [2], propositions 5.3 and 6.4. The proof uses the properties (76)-(80). For the bounds on derivatives one can establish analyticity and use Cauchy bounds.

For the linearization the condition on J insures that there is no contribution from J^χ. There is no contribution from $\tilde{I}^{U - (X_\chi \cup X)}$ since $\chi = \emptyset, X = \emptyset$ is not allowed. The only contribution is from $\check{K}^\#(X)$ and it has the form stated.

4.2

Now we make some further specializations. First for a smooth function $f(\phi)$ on $\phi \in \mathbb{R}^\Lambda$ let $T_2 f$ denote a second order Taylor expansion:

$$(T_2 f)(\phi) = f(0) + f_1(0; \phi) + \frac{1}{2} f_2(0; \phi, \phi) \tag{116}$$

With $K^\#$ defined in (115) we now define for $X \in \mathcal{S}_j$, $X \supset B$, $X \neq B$:

$$J(B, X) = \frac{1}{|X|_j} T_2 K^\#(X) \tag{117}$$

and $J(B, B)$ so (113) is satisfied. Otherwise $J(B, X) = 0$.

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We also specify as in (60) that

$$I(B) = I(E, \sigma, B) = \exp(-V(E, \sigma, B)) \tag{118}$$

and narrow the choice of $\tilde{I}$ by requiring it to have the same form

$$\tilde{I}(B) = I(\tilde{E}, \tilde{\sigma}, B) = \exp(-V(\tilde{E}, \tilde{\sigma}, B)) \tag{119}$$

with $\tilde{E}, \tilde{\sigma}$ still to be specified. Note that since $\sum_{B \subset B'} V(B) = V(B')$ we have that $I'(B') = I(E', \sigma', B') = \exp(-V(E', \sigma', B))$ where

$$E'(B') = \sum_{B \subset B'} \tilde{E}(B) \qquad \sigma' = \tilde{\sigma} \tag{120}$$

Now we have a map $K' = K'(\tilde{E}, \tilde{\sigma}, E, \sigma, K)$. As a norm on the energy we take

$$\|E\|_j = \sup_{B \in \mathcal{B}_j} |E(B)| \tag{121}$$

Then the theorem becomes:

Theorem 4 *Let A be sufficiently large.*

1. *For $R > 0$ there is a $r > 0$ such that the following holds for all j. If $\|\tilde{E}\|_j, |\tilde{\sigma}|, \|E\|_j, |\sigma|, \|K\|_j < r$ then $\|K'\|_{j+1} < R$. Furthermore K' is a smooth function of $\tilde{E}, \tilde{\sigma}, E, \sigma, K$ on this domain with derivatives bounded uniformly in j.*

2. *The linearization of K' at the origin has the form*

$$\mathcal{L}_1 K + \mathcal{L}_2 K + \mathcal{L}_3(E, \sigma, \tilde{E}, \tilde{\sigma}, K) \tag{122}$$

   where

$$\mathcal{L}_1 K(U) = \sum_{X \in \mathcal{P}_{j,c}, X \notin \mathcal{S}_j, \overline{X} = U} K^{\#}(X)$$

$$\mathcal{L}_2 K(U) = \sum_{X \in \mathcal{S}_j, \overline{X} = U} (I - T_2) K^{\#}(X) \tag{123}$$

$$\mathcal{L}_3(E, \sigma, \tilde{E}, \tilde{\sigma}, K)(U) = \sum_{\overline{B} = U} \left(V(\tilde{E}, \tilde{\sigma}, B) - V^{\#}(E, \sigma, B) + \sum_{X \in \mathcal{S}_j, X \supset B} \frac{1}{|X|_j} T_2 K^{\#}(X) \right)$$

Proof. The new map is the composition of the map $K' = K'(I, \tilde{I}, J, K)$ of theorem 3 with the maps $I = I(E, \sigma), \tilde{I} = I(\tilde{E}, \tilde{\sigma}), J = J(K)$. Thus it suffices to establish uniform bounds and smoothness for the latter.

For $I = I(E, \sigma)$ argue as follows. First we note that by (84) there is a constant c such that the function $\sigma \to \exp(-\sigma V(B))$ is analytic in $|\sigma| \leq ch^{-2}$ and satisfies $\|\exp(-\sigma V(B))\|_{s,j} \leq 2$ on that domain. Now if $|\sigma| \leq ch^{-2}/2$ we can write

$$e^{-\sigma V(B)} - 1 = \frac{1}{2\pi i} \int_{|z| = ch^{-2}} \frac{\sigma \, e^{-zV(B)}}{z(z - \sigma)} dz \tag{124}$$

and estimate

$$\|e^{-\sigma V(B)} - 1\|_{s,j} \leq \frac{2|\sigma|}{ch^{-2} - |\sigma|} \leq 4c^{-1}h^2|\sigma| \tag{125}$$

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Hence

$$\|I(E,\sigma,B) - 1\|_{s,j} \leq |e^{-E(B)} - 1| \|e^{-\sigma V(B)}\|_{s,j} + \|e^{-\sigma V(B)} - 1\|_{s,j}$$
$$\leq 2\|E\|_j e^{\|E\|_j} + 4c^{-1}h^2|\sigma| \tag{126}$$

and the same bound holds for $\|I(E,\sigma) - 1\|_{s,j}$. Therefore for any $\epsilon > 0$ there is a $\delta > 0$ such that if $\|E\|_j < \delta$ and $|\sigma| < \delta$ then $\|I(E,\sigma) - 1\|_{s,j} < \epsilon$ for all j. The uniform bounds on derivatives can be verified similarly.

In the same way we show that $\|I(\tilde{E},\tilde{\sigma}) - 1\|'_{s,j}$ can be made uniformly small by bounds on $\|\tilde{E}\|_j$ and $|\tilde{\sigma}|$ with uniform bounds on derivatives.

For the linear map $K \to J$ we first estimate $\|T_2 K^\#(X)\|'_j$. As in the proof of lemma 3 we find that for $n = 0, 1, 2$

$$\frac{1}{n!}\|(T_2 K^\#(X))_n(\phi')\|_j \leq 2\|K^\#(X)\|_j G_{s,j}(X,\phi',0) \tag{127}$$

Summing over n we get a similar bound for $\|T_2 K^\#(X,\phi')\|_j$. Then by (73)

$$\|T_2 K^\#(X)\|'_j \leq \mathcal{O}(1)\|K^\#(X)\|'_j \tag{128}$$

By (79) this is bounded by $\mathcal{O}(1)\|K\|_j$ Then for $X \neq B$ we have $\|J(X,B)\|'_j \leq \|T_2 K^\#(X)\|'_j \leq \mathcal{O}(1)\|K\|_j$. The same bound holds for $\|J(B,B)\|'_j$ and hence $\|J\|'_j \leq \mathcal{O}(1)\|K\|_j$ which which suffices.

The linearization is a computation. Indeed $J(B,X)$ is designed so that

$$\sum_{X \in \mathcal{S}_j, \overline{X}=U} \left(K^\#(X) - \sum_{B \subset X} J(B,X) \right)$$
$$= \sum_{\overline{B}=U} \sum_{X \in \mathcal{S}_j, X \supset B} \frac{1}{|X|_j} T_2 K^\#(X) + \sum_{X \in \mathcal{S}_j, \overline{X}=U} (I - T_2) K^\#(X) \tag{129}$$

which accounts for the presence of these terms. Also the linearization of $(I^\#(B) - 1)$ is $-V^\#(E,\sigma,B)$, and so forth. This completes the proof.

Next we make some estimates on the linearization.

Lemma 5 *Let A be sufficiently large depending on L. Then the operator $\mathcal{L}_1$ is a contraction with a norm which goes to zero as $A \to \infty$.*

Proof. We estimate by (79), (80)

$$\|\mathcal{L}_1 K(U)\|_{j+1} \leq \|\mathcal{L}_1 K(U)\|'_j \leq \sum_{X \notin \mathcal{S}_j, \overline{X}=U} \|K^\#(X)\|'_j \leq \sum_{X \notin \mathcal{S}_j, \overline{X}=U} (A/2)^{-|X|_j} \|K\|_j \tag{130}$$

Multiply by $A^{|U|_{j+1}}$ and take the supremum over U. This yields

$$\|\mathcal{L}_1 K\|_{j+1} \leq \left[\sup_U A^{|U|_{j+1}} \sum_{X \notin \mathcal{S}, \overline{X}=U} (A/2)^{-|X|_j} \right] \|K\|_j \tag{131}$$

The bracketed expression goes to zero as $A \to \infty$ (Brydges [2], lemma 6.18). Thus for A sufficiently large it is arbitrarily small. The idea is that for large polymers X such that $\overline{X} = U$ the quantity $|X|_j$ must dominate $|U|_{j+1}$.

Lemma 6 *Let L be sufficiently large . Then the operator $\mathcal{L}_2$ is a contraction with a norm which goes to zero as $L \to \infty$.*

Proof. This is exactly Brydges [2], proposition 6.11, but to account for some differences in notation and for completeness we include some details. Write $\mathcal{L}_2 K(U) = \sum_{X \in \mathcal{S}_j, \overline{X} = U} R_X(U)$ where $R_X(U) = (I - T_2)K^{\#}(X)$. We have ([2], (6.40))

$$\|R_X(U, \phi)\|_{j+1} \leq \left(1 + \|\phi\|_{\Phi_{j+1}(X^*)}^3\right) \|K_3^{\#}(X, \phi)\|_{j+1} \tag{132}$$

and by (64)

$$\begin{aligned}\|K_3^{\#}(X, \phi)\|_{j+1} &\leq L^{-3d/2}\|K_3^{\#}(X, \phi)\|_j \\ &\leq 3! L^{-3d/2}\|K^{\#}(X, \phi)\|_j \leq 3! L^{-3d/2}\|K^{\#}(X)\|_j' G_j(X, \phi', 0)\end{aligned} \tag{133}$$

and for $\phi = \phi' + \zeta$ ([2], (6.58))

$$\left(1 + \|\phi\|_{\Phi_{j+1}(X^*)}^3\right) G_j(X, \phi, 0) \leq \mathcal{O}(1) G_{j+1}(\bar{X}, \phi', \zeta) \tag{134}$$

Combining these yields

$$\|R_X(U, \phi)\|_{j+1} \leq \mathcal{O}(L^{-3d/2})\|K^{\#}(X)\|_j' G_{j+1}(\bar{X}, \phi', \zeta) \tag{135}$$

and hence using also (79)

$$\|R_X(U)\|_{j+1} \leq \mathcal{O}(L^{-3d/2})\|K^{\#}(X)\|_j' \leq \mathcal{O}(L^{-3d/2})(A/2)^{-|X|_j}\|K\|_j \tag{136}$$

Therefore

$$\|\mathcal{L}_2 K(U)\|_{j+1} \leq \sum_{X \in \mathcal{S}_j, \bar{X} = U} \|R_X(U)\|_{j+1} \leq \mathcal{O}(L^{-3d/2}) \sum_{X \in \mathcal{S}_j, \bar{X} = U} (A/2)^{-|X|_j}\|K\|_j \tag{137}$$

and so

$$\|\mathcal{L}_2 K\|_{j+1} \leq \mathcal{O}(L^{-3d/2}) \left[\sup_U A^{|U|_{j+1}} \sum_{X \in \mathcal{S}_j, \bar{X} = U} (A/2)^{-|X|_j}\right] \|K\|_j \tag{138}$$

But the bracketed expression is $\mathcal{O}(L^d)$ ([2], (6.90)) so we have $\|\mathcal{L}_2 K\|_{j+1} \leq \mathcal{O}(L^{-d/2})\|K\|_j$ to complete the proof.

4.3

The term $\mathcal{L}_3$ needs a more extensive treatment. First we localize the final term in $\mathcal{L}_3$ which is

$$\sum_{\overline{B} = U} \sum_{X \in \mathcal{S}_j, X \supset B} \frac{1}{|X|_j} \left(K^{\#}(X, 0) + \frac{1}{2} K_2^{\#}(X, 0; \phi, \phi)\right) \tag{139}$$

In $K_2^{\#}(X, 0; \phi, \phi)$ pick a point $z \in B$ replace $\phi(x)$ by

$$\phi(z) + \frac{1}{2}(x - z) \cdot \partial\phi(z) \equiv \phi(z) + \frac{1}{2}\sum_{\mu}(x_\mu - z_\mu)\partial_\mu\phi(z) \tag{140}$$

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with the thought that the difference is irrelevant [4]. However $\phi(z)$ and $z \cdot \partial\phi(z)$ are constants and do not contribute. Thus we replace $\phi(x)$ by $\frac{1}{2}x \cdot \partial\phi(z)$. If we also average over $z \in B$ our expression becomes

$$\sum_{\overline{B}=U} \sum_{X \supset B} \frac{1}{|X|_j} \left(K^{\#}(X,0) + \frac{1}{8|B|} \sum_{z \in B} \sum_{\mu\nu} K_2^{\#}(X,0;x_\mu,x_\nu)\partial_\mu\phi(z)\partial_\nu\phi(z) \right) + \mathcal{L}_3'K(U) \qquad (141)$$

where $\mathcal{L}_3'K(U)$ is the error, namely

$$\mathcal{L}_3'K(U) = \sum_{\overline{B}=U} \sum_{X \in \mathcal{S}_j : X \supset B} \frac{1}{|X|_j} \sum_{z \in B} \frac{1}{|B|} \left(\frac{1}{2}K_2^{\#}(X,0;\phi,\phi) - \frac{1}{8}K_2^{\#}(X,0;x \cdot \partial\phi(z), x \cdot \partial\phi(z)) \right) \qquad (142)$$

Next we define

$$\beta(B) = \beta(K,B) = - \sum_{X \in \mathcal{S}_j, X \supset B} \frac{1}{|X|_j} K^{\#}(X,0)$$

$$\alpha_{\mu\nu}(B) = \alpha_{\mu\nu}(K,B) = -\frac{1}{2}\frac{1}{|B|} \sum_{X \in \mathcal{S}_j, X \supset B} \frac{1}{|X|_j} K_2^{\#}(X,0;x_\mu,x_\nu) \qquad (143)$$

Note that $\alpha_{\mu\nu}$ is symmetric and satisfies $\alpha_{-\mu\nu} = -\alpha_{\mu\nu}$. We also let $\alpha_{\mu\nu}$ stand for the function $\alpha_{\mu\nu}(x)$ which takes the constant value $\alpha_{\mu\nu}(B)$ for $x \in B$.

Now we write (141) as

$$-\sum_{\overline{B}=U} \left(\beta(B) + \frac{1}{4} \sum_{z \in B} \sum_{\mu\nu} \alpha_{\mu\nu}(B)\partial_\mu\phi(z)\partial_\nu\phi(z) \right) + \mathcal{L}_3'K(U)$$

$$= -\sum_{\overline{B}=U} V(\beta,\alpha,B,\phi) + \mathcal{L}_3'K(U) \qquad (144)$$

with $V(\beta,\alpha,B,\phi)$ defined as in (59). Altogether then we have

$$\mathcal{L}_3(E,\sigma,\tilde{E},\tilde{\sigma},K)(U) = \sum_{\overline{B}=U} \left(V(\tilde{E},\tilde{\sigma},B) - V^{\#}(E,\sigma,B) - V(\beta,\alpha,B) \right) + \mathcal{L}_3'K(U) \qquad (145)$$

Lemma 7

$$\|\beta\|_j \equiv \sup_{B \in \mathcal{B}_j} |\beta(B)| \leq \mathcal{O}(1)A^{-1}\|K\|_j$$

$$\|\alpha\|_j \equiv \sup_{B \in \mathcal{B}_j} \sum_{\mu\nu} |\alpha_{\mu\nu}(B)| \leq \mathcal{O}(1)h^{-2}A^{-1}\|K\|_j \qquad (146)$$

Remark. Note that the norm $\|\alpha\|_j$ agrees with the norm $\|s\|_j$ in (82) if $s_{\mu\nu}(x) = \alpha_{\mu\nu}(B)$ for $x \in B$.

Proof. By (79) we have

$$|K^{\#}(X,0)| \leq \|K^{\#}(X)\|_j' \leq (A/2)^{-1}\|K\|_j$$

$$\|K_2^{\#}(X,0)\|_j \leq 2\|K^{\#}(X)\|_j' \leq A^{-1}\|K\|_j \qquad (147)$$

Since the number of small sets containing a block B is bounded by a constant depending only on the dimension we have

$$|\beta(B)| \leq \sum_{X \in \mathcal{S}_j, X \supset B} |K^{\#}(X,0)| \leq \mathcal{O}(1)A^{-1}\|K\|_j \qquad (148)$$

---

[4]We need the factor $1/2$ since the sum is over $\pm\mu = 1, \ldots, d$. The convention is that $x_{-\mu} = -x_\mu$

For the bound on α note that $\|x_\mu\|_{\Phi_j(X^*)} = h^{-1}L^{dj/2}$. Then since $|B|^{-1} = L^{-dj}$

$$|B|^{-1}|K_2^\#(X,0;x_\mu,x_\nu)| \le h^{-2}\|K_2^\#(X,0)\|_j \le h^{-2}A^{-1}\|K\|_j \tag{149}$$

whence

$$\sum_{\mu\nu}|\alpha_{\mu\nu}(B)| \le \sum_{\mu\nu}\sum_{X\in\mathcal{S}_j,X\supset B}|B|^{-1}|K_2^\#(X,0;x_\mu,x_\nu)| \le \mathcal{O}(1)h^{-2}A^{-1}\|K\|_j \tag{150}$$

which gives the result

Lemma 8 *Let L be sufficiently large. Then the operator $\mathcal{L}_3'$ is a contraction with arbitrarily small norm.*

Proof. We have

$$\mathcal{L}_3'K(U) = \sum_{\bar{B}=U}\sum_{X\in\mathcal{S}_j:X\supset B}\frac{1}{|X|_j}\sum_{z\in B}\frac{1}{|B|}\left(\frac{1}{2}K_2^\#(X,0;\phi-\frac{1}{2}x\cdot\partial\phi(z),\phi)\right) + \text{ similar} \tag{151}$$

Since $\partial_{-\mu}\phi(x) = -\partial_\mu\phi(x-e_\mu)$ we have

$$\frac{\partial}{\partial x_\mu}\left(\phi(x) - \frac{1}{2}\sum_\nu x_\nu\partial_\nu\phi(z)\right) = \partial_\mu\phi(x) - \frac{1}{2}\partial_\mu\phi(z) - \frac{1}{2}\partial_\mu\phi(z-e_\mu) \tag{152}$$

The same holds with ∂_μ replaced by $\nabla_{j,\mu}$ and then with $\text{diam}_j(X^*) = L^{-j}\,\text{diam}(X^*)$

$$\|\nabla_j(\phi-\frac{1}{2}x\cdot\partial\phi(z))\|_{X^*,\infty} \le \text{diam}_j(X^*)\|\nabla_j^2\phi\|_{X^*,\infty} \tag{153}$$

But $\text{diam}_j(X^*) \le \mathcal{O}(1)$ since X is a small set. Hence

$$\begin{aligned}\|\phi-\frac{1}{2}x\cdot\partial\phi(z)\|_{\Phi_j(X^*)} &\le \mathcal{O}(1)h_j^{-1}\|\nabla_j^2\phi\|_{X^*,\infty}\\ &\le \mathcal{O}(L^{-d/2-1})h_{j+1}^{-1}\|\nabla_{j+1}^2\phi\|_{X^*,\infty}\\ &\le \mathcal{O}(L^{-d/2-1})\|\phi\|_{\Phi_{j+1}(X^*)}\end{aligned} \tag{154}$$

Now we estimate

$$H_X(U,\phi) = K_2^\#(X,0;\phi-\frac{1}{2}x\cdot\partial\phi(z),\phi) \tag{155}$$

We claim that

$$\begin{aligned}|(H_X(U))_0(\phi)| &\le \mathcal{O}(L^{-d-1})\|K_2^\#(X,0)\|_j\|\phi\|_{\Phi_{j+1}(U^*)}^2\\ \|(H_X(U))_1(\phi)\|_{j+1} &\le \mathcal{O}(L^{-d-1})\|\|K_2^\#(X,0)\|_j\|\phi\|_{\Phi_{j+1}(U^*)}\\ \|(H_X(U))_2(\phi)\|_{j+1} &\le \mathcal{O}(L^{-d-1})\|K_2^\#(X,0)\|_j\end{aligned} \tag{156}$$

For example the second bound follows from (64) and (154) by

$$\begin{aligned}&|(H_X(U))_1(\phi;f)|\\ =&|K_2^\#(X,0;\phi-\frac{1}{2}x\cdot\partial\phi(z),f) + K_2^\#(X,0;f-\frac{1}{2}x\cdot\partial f(z),\phi)|\\ \le&\|K_2^\#(X,0)\|_j\left(\|\phi-\frac{1}{2}x\cdot\partial\phi(z)\|_{\Phi_j(X^*)}\|f\|_{\Phi_j(X^*)} + \|f-\frac{1}{2}x\cdot\partial f(z)\|_{\Phi_j(X^*)}\|\phi\|_{\Phi_j(X^*)}\right)\\ \le&\mathcal{O}(L^{-d-1})\|K_2^\#(X,0)\|_j\|\phi\|_{\Phi_{j+1}(X^*)}\|f\|_{\Phi_{j+1}(X^*)}\end{aligned} \tag{157}$$

To complete the bound we need $\|f\|_{\Phi_{j+1}(X^*)} \le \|f\|_{\Phi_{j+1}(U^*)}$ which holds provided $X^* \subset U^*$. Here X^* is an $\mathcal{S}_j$ neighborhood of $X \in \mathcal{S}_j$ and U^* is an $\mathcal{S}_{j+1}$ neighborhood of $U \in \mathcal{B}_{j+1}$.

To see that $X^* \subset U^*$ note first that $X^* \cap U \ne \emptyset$ since both contain B. Suppose $X^* \subset U^*$ is false. Since points not in U^* are separated from points in U by at least L^{j+1} we have $\mathrm{diam}(X^*) \ge L^{j+1}$. On the other hand $\mathrm{diam}(X) \le \mathcal{O}(1)L^j$ so $\mathrm{diam}(X^*) \le \mathcal{O}(1)L^j$. This is a contradiction for L sufficiently large.

Combining these estimates (156) we get

$$\|H_X(U,\phi)\|_{j+1} \le \mathcal{O}(L^{-d-1})\|K_2^\#(X,0)\|_j(1 + \|\phi\|_{\Phi_{j+1}(U^*)}^2) \tag{158}$$

But for $\phi = \phi' + \zeta$

$$(1 + \|\phi\|_{\Phi_{j+1}(U^*)}^2) \le G_{s,j+1}(U,\phi,0) \le G_{s,j+1}(U,\phi',\zeta) \le G_{j+1}(U,\phi',\zeta) \tag{159}$$

Using also (147) we obtain
$$\|H_X(U)\|_{j+1} \le \mathcal{O}(L^{-d-1})A^{-1}\|K\|_j \tag{160}$$

which implies
$$\|\mathcal{L}_3'K(U)\|_{j+1} \le \mathcal{O}(1) \sum_{\bar{B}=U} \|H_X(U)\|_{j+1} \le \mathcal{O}(L^{-1})A^{-1}\|K\|_j \tag{161}$$

Since $\mathcal{L}_3'K(U)$ is zero unless $|U|_{j+1} = 1$ this gives $\|\mathcal{L}_3'K\|_{j+1} \le \mathcal{O}(L^{-1})\|K\|_j$ which completes the proof.

4.4

Now consider the first term in (145). We would like to choose $\tilde{E}, \tilde{\sigma}$ so it vanishes but are not quite there yet.

To proceed we add another hypothesis. We assume that $E(B), K(X,\phi)$ are invariant under lattice symmetries for B, X away from the boundary of Λ_N, that is if B, X have no boundary blocks. More precisely $E(B)$ is independent of B, and if g is a translation, rotation by a multiple of $\pi/2$, or a reflection and $(g\phi)(x) = \phi(g^{-1}x)$ then $K(gX, g\phi) = K(X,\phi)$ provided X, gX are away from the boundary.

These properties carry over to the next level and to the quantities $\beta(B), \alpha_{\mu\nu}(B)$.

Lemma 9 *Suppose $E(B), K(X,\phi)$ are invariant under lattice symmetries away from the boundary of Λ_N and $\tilde{E}(B)$ is invariant for B^* away from the boundary. Then*

1. *$E'(B'), K'(U,\phi)$ are invariant for B', U away from the boundary*

2. *If B^* is away from the boundary then $\beta(B), \alpha_{\mu\nu}(B)$ are independent of B and $\alpha_{\mu\nu}(B) = \hat{\alpha}_{\mu\nu}(B)$ defined for all B by*

$$\hat{\alpha}_{\mu\nu}(B) = \frac{\alpha}{2} \left(\delta_{\mu\nu} - \delta_{\mu,-\nu}\right) \tag{162}$$

where α is a constant.

Proof. If $B' \in \mathcal{B}_{j+1}$ is separated from the boundary then $d(B', \partial\Lambda_N) \ge L^{j+1}$. If $B \subset B'$ then $d(B^*, \partial\Lambda_N) \ge L^{j+1} - 2^d \ge L^j$ so B^* is away from the boundary. Thus in $E'(B') = \sum_{B \subset B'} \tilde{E}(B)$ each $\tilde{E}(B)$ is invariant and hence so is $E'(B')$.

Under our hypotheses $\tilde{K}(X)$ defined with (118), (119) is invariant for X^* away from the boundary, and using the invariance of Γ_j the quantity $J(B,X)$ defined by (117) is invariant for B^* away from the boundary. Thus $\check{K}(X)$ is invariant for X^* away from the boundary and so is $\check{K}^\#(X)$. Now in the

definition (110) of $K'(U)$ the quantity $\check{K}^{\#}(X)$ only contributes for $X \subset U$. Then U away from the boundary implies X^* away from the boundary, so only invariant terms $\check{K}^{\#}(X)$ contribute. Similarly only invariant terms contribute to J^χ and $\tilde{I}^{U-(X_\chi \cup X)}$. Hence $K'(U)$ is invariant.

The quantities $\beta(B), \alpha_{\mu\nu}(B)$ depend on $K(X)$ for $X \subset B^*$ so if B^* is away from the boundary they are invariant and in particular independent of B. Furthermore under the same condition if R is a rotation or a reflection we have for $\mu, \nu > 0$

$$\alpha_{\mu\nu}(B) = \sum_{\mu'\nu'>0} R_{\mu\mu'} \, R_{\nu\nu'} \, \alpha_{\mu'\nu'}(B) \tag{163}$$

To establish the identity $\alpha_{\mu\nu}(B) = \hat{\alpha}_{\mu\nu}(B)$ note that since both are symmetric and satisfy $\alpha_{-\mu\nu}(B) = -\alpha_{\mu\nu}(B)$ it suffices to establish the identity for $\mu, \nu > 0$ in which case it says $\alpha_{\mu\nu}(B) = \alpha \delta_{\mu\nu}/2$. Specializing (163) to reflections through planes $x_\mu = 0$ we deduce that $\alpha_{\mu\nu}(B)$ equals zero unless $\mu = \nu$ so $\alpha_{\mu\nu}(B) = \alpha_\mu \delta_{\mu\nu}/2$. Specializing (163) to rotations we deduce that α_μ is independent of μ and obtain the result. This completes the proof.

We also define for all $B \in \mathcal{B}_j$
$$\alpha'_{\mu\nu}(B) = \alpha \, \delta_{\mu\nu} \tag{164}$$

and write for any $U \in \mathcal{B}_{j+1}$

$$\sum_{\overline{B}=U} V(\beta, \alpha, B) = \sum_{\overline{B}=U} V(\beta, \alpha', B) - \mathcal{L}_4 K(U) - \Delta K(U) \tag{165}$$

where for $U \subset \mathcal{B}_{j+1}$

$$\mathcal{L}_4 K(U) = \sum_{\overline{B}=U} V(0, \alpha' - \hat{\alpha}, B) = V(0, \alpha' - \hat{\alpha}, U)$$
$$\Delta K(U) = \sum_{\overline{B}=U} V(0, \hat{\alpha} - \alpha, B) = V(0, \tilde{\alpha}, U) \tag{166}$$

where $\tilde{\alpha}_{\mu\nu}(x) = \hat{\alpha}_{\mu\nu}(B) - \alpha_{\mu\nu}(B)$ if $x \in B$. Note that $\Delta K(U)$ vanishes unless U touches the boundary. Now (145) becomes

$$\mathcal{L}_3(E, \sigma, \tilde{E}, \tilde{\sigma}, K)(U)$$
$$= \sum_{\overline{B}=U} \left(V(\tilde{E}, \tilde{\sigma}, B) - V^{\#}(E, \sigma, B) - V(\beta, \alpha', B) \right) + \mathcal{L}'_3 K(U) + \mathcal{L}_4 K(U) + \Delta K(U) \tag{167}$$

Lemma 10 *Let L be sufficiently large. Then the operator $\mathcal{L}_4$ is a contraction with arbitrarily small norm.*

Proof. For $U \in \mathcal{B}_{j+1}$
$$\mathcal{L}_4 K(U) = \frac{\alpha}{8} \sum_\mu \sum_{x \in U} \partial_\mu \phi(x) \, \partial_{-\mu} \phi(x) + \partial_\mu \phi(x)^2 \tag{168}$$

But $\partial_{-\mu}\phi(x) = -\partial_\mu\phi(x - e_\mu)$ and $\partial_\mu\phi(x - e_\mu) - \partial_\mu\phi(x) = -(\partial_{-\mu}\partial_\mu\phi)(x)$ so this is

$$\mathcal{L}_4 K(U) = -\frac{\alpha}{8} \sum_\mu \sum_{x \in U} (\partial_{-\mu}\partial_\mu\phi)(x)\partial_\mu\phi(x) \tag{169}$$

The proof now proceeds as in lemma 3 but now on scale $j+1$. Instead of (87) we have

$$
\begin{aligned}
|\partial_\mu \phi(x)| &\leq hL^{-d(j+1)/2}\|\phi\|_{\Phi_{j+1}(U^*)} \\
|\partial_{-\mu}\partial_\mu \phi(x)| &\leq hL^{-d(j+1)/2}L^{-(j+1)}\|\phi\|_{\Phi_{j+1}(U^*)}
\end{aligned}
\tag{170}
$$

The factor $L^{-d(j+1)}$ compensates the sum over $x \in U$ and taking $L^{-(j+1)} \leq L^{-1}$ one obtains for the strong norm and hence the weak norm

$$
\|\mathcal{L}_4 K(U)\|_{j+1} \leq \mathcal{O}(L^{-1})h^2|\alpha|
\tag{171}
$$

However $|\alpha| \leq \mathcal{O}(1)h^{-2}A^{-1}\|K\|_j$ by lemma 7 which yields $\|\mathcal{L}_4 K\|_{j+1} \leq \mathcal{O}(L^{-1})\|K\|_j$.

Lemma 11 *Let L be sufficiently large. Then the operator Δ is a contraction with arbitrarily small norm.*

Proof. By lemma 3

$$
\|\Delta K(U)\|_{j+1} = \|V(0,\tilde{\alpha},U)\|_{j+1} \leq h^2\|\tilde{\alpha}\|_{j+1} = h^2 \sup_{U \in \mathcal{B}_{j+1}} L^{-(j+1)d}\|\tilde{\alpha}\|_{1,U}
\tag{172}
$$

But $\tilde{\alpha}(x) = 0$ if $x \in B$ and B^* is away from the boundary. Hence it vanishes if $d(x,\partial\Lambda_N) > 2^d L^j$ and so

$$
\|\tilde{\alpha}\|_{1,U} \leq \mathcal{O}(1)|\alpha|\big|\{x \in U : d(x,\partial\Lambda_N) \leq 2^d L^j\}\big| \leq \mathcal{O}(L^{(j+1)(d-1)}L^j)|\alpha|
\tag{173}
$$

Combining these with $|\alpha| \leq \mathcal{O}(1)h^{-2}A^{-1}\|K\|_j$ we obtain $\|\Delta K(U)\|_{j+1} \leq \mathcal{O}(L^{-1})A^{-1}\|K\|_j$ and hence $\|\Delta K\|_{j+1} \leq \mathcal{O}(L^{-1})\|K\|_j$.

4.5

We now choose $\tilde{E}(B), \tilde{\sigma}$ so the V terms in (167) cancel. First note that

$$
\begin{aligned}
V^\#(E,\sigma,B,\phi) &= E(B) + \int \frac{\sigma}{4}\sum_{x\in B}\sum_\mu (\partial_\mu\phi(x) + \partial_\mu\zeta(x))^2 d\mu_{\Gamma_{j+1}}(\zeta) \\
&= E(B) + \frac{\sigma}{4}\sum_{x\in B}\sum_\mu \partial_\mu\phi(x)^2 + \frac{\sigma}{4}\sum_{x\in B}\sum_\mu (\partial_\mu\Gamma_{j+1}\partial_\mu^*)(x,x) \\
&\equiv V(E,\sigma,B,\phi) + \frac{\sigma}{4}\sum_\mu Tr(1_B\partial_\mu\Gamma_{j+1}\partial_\mu^*)
\end{aligned}
\tag{174}
$$

Thus the constant terms cancel if we define $\tilde{E} = \tilde{E}(E,\sigma,K)$ by

$$
\tilde{E}(B) = E(B) + \frac{\sigma}{4}\sum_\mu Tr(1_B\partial_\mu\Gamma_{j+1}\partial_\mu^*) + \beta(K,B)
\tag{175}
$$

The second order terms vanish if we define $\tilde{\sigma} = \tilde{\sigma}(\sigma,K)$ by

$$
\tilde{\sigma} = \sigma + \alpha(K)
\tag{176}
$$

Note that we are canceling the constant term exactly for all B, but for the quadratic term we are only canceling exactly the invariant version away from the boundary.

By composing $K' = K'(\tilde{E},\tilde{\sigma},E,\sigma,K)$ with $\tilde{E} = \tilde{E}(E,\sigma,K)$ and $\tilde{\sigma} = \tilde{\sigma}(\sigma,K)$ we obtain a new map $K' = K'(E,\sigma,K)$. We also have new quantities $E'(E,\sigma,K)$ defined by $E'(B') = \sum_{B\subset B'}\tilde{E}(B)$ and $\sigma' = \sigma'(\sigma,K)$ defined by $\sigma' = \tilde{\sigma} = \sigma + \alpha(K)$. These quantities satisfy (c.f. (104))

$$
\mu_{\Gamma_{j+1}} * \left(I(E,\sigma)\circ K\right)(\Lambda) = \left(I'(E',\sigma')\circ K'\right)(\Lambda)
\tag{177}
$$

We continue to assume that L is sufficiently large, and that A is sufficiently large depending on L.

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Theorem 5

1. *For $R > 0$ there is a $r > 0$ such that the following holds for all j. If $\|E\|_j, |\sigma|, \|K\|_j < r$ then $\|E'\|_{j+1}, |\sigma'|, \|K'\|_{j+1} < R$. Furthermore E', K', σ' are smooth functions of E, σ, K on this domain with derivatives bounded uniformly in j.*

2. *The linearization of $K' = K'(E, \sigma, K)$ at the origin is the contraction $\mathcal{L}K$ where*

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3' + \mathcal{L}_4 + \Delta \tag{178}$$

Proof. For the first part it suffices to show that the linear maps $\tilde{E} = \tilde{E}(E, \sigma, K)$ and $\tilde{\sigma} = \tilde{\sigma}(\sigma, K)$ have norms bounded uniformly in j. The bound on $\tilde{\sigma}$ follows from $|\alpha(K)| \leq \mathcal{O}(1)h^{-2}A^{-1}\|K\|_j$ from lemma 7. The bound on $\tilde{E}$ follows from the bound on $\|\beta(K)\|_j \leq \mathcal{O}(1)A^{-1}\|K\|_j$ from lemma 7 and the estimate (42) which gives for $B \in \mathcal{B}_j$

$$\left| \frac{\sigma}{4} \sum_\mu Tr(1_B(\partial_\mu \Gamma_j \partial_\mu^*)) \right| \leq \mathcal{O}(1)|\sigma| \sum_{x \in B} L^{-dj} \leq \mathcal{O}(1)|\sigma| \tag{179}$$

Together they imply that $\tilde{E} = \tilde{E}(E, \sigma, K)$ satisfies

$$\|\tilde{E}\|_j \leq \|E\|_j + \mathcal{O}(1)(|\sigma| + A^{-1}\|K\|_j) \tag{180}$$

The second part follows since the linearization of the new function K' is the linearization of the old function K' composed with $\tilde{E} = \tilde{E}(E, \sigma, K), \tilde{\sigma} = \tilde{\sigma}(\sigma, K)$. (All vanish at zero.) This effects the cancellation and leaves us with $\mathcal{L}K$.

4.6

It is convenient to decouple the energy from the other variables. Suppose we start with $E(B) = 0$ in (177). Then

$$\mu_{\Gamma_{j+1}} * (I(0, \sigma) \circ K)(\Lambda_N) = (I'(E', \sigma') \circ K')(\Lambda) \tag{181}$$

where $\sigma' = \sigma'(\sigma, K)$ and $K' = K'(0, \sigma, K)$ and $E' = E'(0, \sigma, K)$. Next remove the E' making an adjustment in K'. We relabel everything with a plus and write

$$\mu_{\Gamma_{j+1}} * (I(0, \sigma) \circ K)(\Lambda_N) = \exp\left(\sum_{B' \in \mathcal{B}_{j+1}(\Lambda_N)} E^+(B') \right) (I'(0, \sigma^+) \circ K^+)(\Lambda_N) \tag{182}$$

where

$$E^+(\sigma, K, B') \equiv E'(0, \sigma, K, B') = \sum_{B \subset B'} \tilde{E}(0, \sigma, K, B) \qquad B' \in \mathcal{B}_{j+1}$$

$$\sigma^+(\sigma, K) \equiv \sigma'(\sigma, K) = \sigma + \alpha(K) \tag{183}$$

$$K^+(\sigma, K, U) \equiv \exp\left(-\sum_{B' \subset U} E^+(B') \right) K'(0, \sigma, K, U) \qquad U \in \mathcal{P}_{j+1}$$

The dynamical variables are now $\sigma^+(\sigma, K)$ and $K^+(\sigma, K)$. The energy $E^+(\sigma, K)$ is driven by the other variables. Since everything vanishes at the origin the linearization of $K^+(\sigma, K)$ is still $\mathcal{L}K$. The bound (180) on $\tilde{E}$ gives a bound on E^+ and our main theorem becomes:

Theorem 6

1. *For $R > 0$ there is a $r > 0$ such that the following holds for all j. If $|\sigma|, \|K\|_j < r$ then $|\sigma^+|, \|K^+\|_{j+1} < R$. Furthermore σ^+, K^+ are smooth functions of σ, K on this domain with derivatives bounded uniformly in j.*

2. *The extracted energies satisfy*

$$\|E^+(\sigma, K)\|_{j+1} \le \mathcal{O}(L^d)\Big(|\sigma| + A^{-1}\|K\|_j\Big) \tag{184}$$

3. *The linearization of K^+ at the origin is the contraction $\mathcal{L}$.*

5 The stable manifold

Now we etablish the existence of a stable manifold for the flow. For now we do not specialize to the dipole gas, but take a general initial point σ_0, K_0 corresponding to an integral $\int (I(0, \sigma_0) \circ K_0)(\Lambda_N) d\mu_{C_0}$.

We assume $K_0(X, \phi)$ has the lattice symmetries and satisfies the conditions (61). We also assume $|\sigma_0|, \|K_0\|_0 < r$ where r is small enough so the last theorem holds, say with $R = 1$, and we can take the first step. We apply the transformation (182) for $j = 0, 1, 2, \dots$ and continue as long as we can. This generates a sequence $\sigma_j, K_j^N(X)$ by $\sigma_{j+1} = \sigma^+(\sigma_j, K_j^N)$ and $K_{j+1}^N = K^+(\sigma_j, K_j^N)$ with extracted energies $E_{j+1}^N = E^+(\sigma_j, K_j^N)$. Then we have with $I_j(\sigma_j) = I_j(0, \sigma_j)$ for any k

$$\int (I_0(\sigma_0) \circ K_0)(\Lambda_N) d\mu_{C_0} = \exp\left(\sum_{j=1}^k \sum_{B \in \mathcal{B}_j(\Lambda_N)} E_j^N(B)\right) \int (I_k(\sigma_k) \circ K_k^N)(\Lambda_N) d\mu_{C_k} \tag{185}$$

The quantities $K_j^N(X)$ and $E_j^N(B)$ are independent of N and have the lattice symmetries if X, B are away from $\partial \Lambda_N$ in the sense that they have no boundary blocks. These properties are true initially and are preserved by the iteration. In this case we denote these quantities by just $K_j(X)$ and $E_j(B)$

By our construction α defined in (143),(162) only depends on K_j and splitting K^+ into a linear and a higher order piece the sequence $\sigma_j, K_j^N(X)$ is generated by the RG transformation

$$\begin{aligned} \sigma_{j+1} &= \sigma_j + \alpha(K_j) \\ K_{j+1}^N &= \mathcal{L}(K_j^N) + f(\sigma_j, K_j^N) \end{aligned} \tag{186}$$

This is regarded as a mapping from the Banach space $\mathbb{R} \times \mathcal{K}_j(\Lambda_N)$ to the Banach space $\mathbb{R} \times \mathcal{K}_{j+1}(\Lambda_N)$ The function $f = f_j$ is smooth with derivatives bounded uniformly in j and satisfies $f(0,0) = 0$, $Df(0,0) = 0$.

For this mapping we can use the stable manifold theorem proved in Brydges [2] to obtain:

Theorem 7 *Let L be sufficiently large, A sufficiently large (depending on L), and r sufficiently small (depending on L, A). Then there is $0 < \rho < r$ and a smooth real-valued function $\sigma_0 = h(K_0)$, $h(0) = 0$, mapping $\|K_0\|_0 < \rho$ into $|\sigma_0| < r$ such that with these start values the sequence σ_j, K_j^N is defined for all $0 \le j \le N$ and*

$$|\sigma_j| \le r2^{-j} \qquad \|K_j^N\|_j \le r2^{-j} \tag{187}$$

Furthermore the extracted energies satisfy

$$\|E_{j+1}^N\|_{j+1} \le \mathcal{O}(L^d) r 2^{-j} \tag{188}$$

Proof. We first establish the theorem for the invariant quantities $K_j(X), E_j(B)$ away from the boundary. In this case the RG transformation (186) can be regarded as a map from the Banach space $\mathbb{R} \times \mathcal{K}_j(\mathbb{Z}^d)$ to the Banach space $\mathbb{R} \times \mathcal{K}_{j+1}(\mathbb{Z}^d)$, since any $X \in \mathcal{P}_{j,c}(\mathbb{Z}^d)$ is well inside Λ_N for N sufficiently large. On this space the transformation can be iterated indefinitely. Furthermore $\mathcal{L}$ has the form $\mathcal{L} = \mathcal{L}' + \Delta$ where $\mathcal{L}' = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}'_3 + \mathcal{L}_4$ and where Δ vanishes away from the boundary. Thus the RG transformation on the invariant quantities is

$$
\begin{aligned}
\sigma_{j+1} &= \sigma_j + \alpha(K_j) \\
K_{j+1} &= \mathcal{L}'(K_j) + f(\sigma_j, K_j)
\end{aligned}
\tag{189}
$$

Both $\mathcal{L}', \alpha$ are contractions with arbitrarily small norm for A, L large. Then we can apply the stable manifold theorem from [2], Theorem 2.16 with parameters $\mu = 1/2$ and $\alpha = 1$. This yields the function $\sigma_0 = h(K_0)$ and with these initial values the sequence σ_j, K_j satisfies (189) with the bounds (187).

Once we know that σ_j is not growing we can give a direct proof that $\|K_j^N\|_j$ satisfies the bound (187) reproducing the results for K_j but now including the boundary polymers. The bound is true initially since $K_0^N(X) = K_0(X)$ even if X touches the boundary. Suppose it is true for j. We have $K_{j+1}^N = \mathcal{L}(K_j^N) + f(\sigma_j, K_j^N)$ where $\mathcal{L}$ is a contraction with norm less than $1/4$ and $f(\sigma_j, K_j^N)$ is second order. Hence for some constant M and r sufficiently small

$$
\begin{aligned}
\|K_{j+1}^N\|_{j+1} &\leq \frac{1}{4}\|K_j^N\|_j + M\left(|\sigma_j|^2 + \|K_j^N\|_j^2\right) \\
&\leq \frac{1}{4}\left(r2^{-j}\right) + 2M\left(r2^{-j}\right)^2 \\
&\leq r2^{-j-1}
\end{aligned}
\tag{190}
$$

which is the bound for $j + 1$.

Finally the energy bound (188) comes from the bounds on σ_j, K_j^N and (184).

6 The dipole gas

6.1 the initial density

We now specialize to the dipole gas and complete the proof of the theorem. The first issue is to adjust the dipole gas density so it becomes a point on the stable manifold.

For the dipole gas the initial density $\mathcal{Z}_0^N = \mathcal{Z}_0^N(z, \sigma)$ is given in (49). We break it into pieces defining for $B \in \mathcal{B}_0$, $W_0(B) = zW(\sqrt{1 + \sigma_0}, B)$ as in (92) and $V_0(B) = \sigma_0 V(B)$ as in (60). Then we follow with a Mayer expansion to put the density in the form we want.

$$
\begin{aligned}
\mathcal{Z}_0^N &= \prod_{B \subset \Lambda_N} e^{W_0(B) - V_0(B)} = \prod_{B \subset \Lambda_N} \left(e^{-V_0(B)} + (e^{W_0(B)} - 1)e^{-V_0(B)}\right) \\
&= \sum_{X \subset \Lambda_N} I_0(\sigma_0, \Lambda_N - X)K_0(X) = (I_0(\sigma_0) \circ K_0)(\Lambda_N)
\end{aligned}
\tag{191}
$$

where $I_0(\sigma_0, B) = e^{-V_0(B)}$ and $K_0(X) = K_0(z, \sigma_0, X)$ is given by

$$
K_0(X) = \prod_{B \subset X} (e^{W_0(B)} - 1)e^{-V_0(B)}
\tag{192}
$$

Note that K_0 has the lattice symmetries and satisfies the conditions (61). To start the flow we need:

Lemma 12 *Given $r > 0$ if if $|z|$ and $|\sigma_0|$, are sufficiently small then $\|K_0(z, \sigma_0)\|_0 \leq r$. Furthermore K_0 is a smooth function of (z, σ_0).*

Proof. Consider the $G = 1$ norm $\| \cdot \|_{00}$ defined in (101). As in (102) we have

$$\|e^{W_0(B)} - 1\|_{00} \leq \exp\left(2|z|e^{h\sqrt{d(1+\sigma_0)}}\right) - 1 \leq c|z| \tag{193}$$

for some constant c. Also by lemma 3 $\|e^{-V_0(B)}\|_{s,0} \leq 2$. Combining these

$$\|(e^{W_0(B)} - 1)e^{-V_0(B)}\|_{s,0} \leq \|e^{W_0(B)} - 1\|_{00}\|e^{-V_0(B)}\|_{s,0} \leq 2c|z| \tag{194}$$

Then

$$\|K_0(X)\|_{s,0} \leq \prod_{B \subset X} \|(e^{W_0(B)} - 1)e^{-V_0(B)}\|_{s,0} \leq (2c|z|)^{|X|_0} \tag{195}$$

Then same follows for the weak norm $\|K_0(X)\|_0$ and so

$$\|K_0\|_0 = \sup_{X \in \mathcal{P}_{0,c}} \|K_0(X)\|_0 A^{|X|_0} \leq \sup_X (2c|z|A)^{|X|_0} \leq 2c|z|A < r \tag{196}$$

The smoothness follows similarly from lemma 3 and lemma 4. For example consider the part of K_0 depending on W which is

$$K_0'(X) = \prod_{B \subset X} (e^{W_0(B)} - 1) \tag{197}$$

We show that the derivative with respect to σ_0 has a finite norm. The derivative is computed as

$$\frac{\partial K_0'(X)}{\partial \sigma_0} = \sum_{B_0 \subset X} zW'(\sqrt{1+\sigma_0}, B)\frac{1}{2\sqrt{1+\sigma_0}} \prod_{B \subset X - B_0} (e^{W_0(B)} - 1) \tag{198}$$

Then by (98) and (193) we have for some constant c'

$$\|\frac{\partial K_0'(X)}{\partial \sigma_0}\|_{s,0} \leq \sum_{B_0 \subset X} |z|\|W'(\sqrt{1+\sigma_0}, B)\|_{s,0} \prod_{B \subset X - B_0} \|e^{W_0(B)} - 1\|_{00} \leq (c'|z|)^{|X|_0} \tag{199}$$

and so

$$\|\frac{\partial K_0'}{\partial \sigma_0}\|_0 \leq Ac'|z| \tag{200}$$

The other pieces may be treated similarly. [5] This completes the proof.

To apply theorem 7 we need to choose σ_0 so that $\sigma_0 = h(K_0(z, \sigma_0))$.

Lemma 13 *The equation $\sigma = h(K_0(z, \sigma))$ defines a smooth implicit function $\sigma = \sigma(z)$ near the origin which satisfies $\sigma(0) = 0$.*

Proof. Let $f(z, \sigma) = \sigma - h(K_0(z, \sigma))$. Then $f(0, 0) = 0$. The function h is smooth by theorem 7 and the function K_0 is smooth by lemma 12. Hence f is smooth and we compute

$$f_\sigma(0, 0) = 1 - Dh(0; (K_0)_\sigma(0, 0)) \tag{201}$$

But $K_0(0, \sigma) = 0$, hence $(K_0)_\sigma(0, 0) = 0$ and hence $f_\sigma(0, 0) = 1 \neq 0$. By the implicit function theorem there exists $\sigma = \sigma(z)$ so that $f(z, \sigma(z)) = 0$. This completes the proof.

---

[5] For $\partial K_0/\partial \sigma_0$ we must combine the estimate (199) with estimates $\|e^{-V_0(B)}\|_{s,0} \leq 2$. For this use $G_{s,0}^2 \leq G_0$.

Taking $|z|$ sufficiently small and making the choice $\sigma_0 = \sigma(z)$ the start density $I_0(\sigma(z)) \circ K_0(z, \sigma(z))$ is now tuned and we can apply theorem 7. We have for $0 \le k \le N$

$$Z_N'(z, \sigma(z)) = \int \Big(I_0(\sigma(z)) \circ K_0(z, \sigma(z)) \Big)(\Lambda_N) d\mu_{C_0}$$
$$= \exp\left(\sum_{j=1}^{k} \sum_{B \in \mathcal{B}_j(\Lambda_N)} E_j^N(B) \right) \int (I_k(\sigma_k) \circ K_k^N)(\Lambda_N) d\mu_{C_k} \tag{202}$$

where $|\sigma_j| \le r2^{-j}$ and $\|K_j^N\|_j \le r2^{-j}$ and $\|E_{j+1}^N\|_{j+1} \le \mathcal{O}(L^d) r2^{-j}$.

6.2   the pressure

Now we can show the pressure has an infinite volume limit, completing the proof of theorem 1.

Theorem 8 *For $|z|$ sufficiently small the following limit exists:*

$$\lim_{N \to \infty} |\Lambda_N|^{-1} \log Z_N'(z, \sigma(z)) \tag{203}$$

Proof. Take $k = N$ in (202). At this level there is only one block $\Lambda_N \in \mathcal{B}_N(\Lambda_N)$ and so

$$|\Lambda_N|^{-1} \log Z_N'(z, \sigma(z)) = |\Lambda_N|^{-1} \sum_{j=1}^{N} \sum_{B \in \mathcal{B}_j(\Lambda_N)} E_j^N(B)$$
$$+ |\Lambda_N|^{-1} \log \left(\int \left[I_N(\sigma_N, \Lambda_N) + K_N^N(\Lambda_N) \right] d\mu_{C_N} \right) \tag{204}$$

The second term has the form

$$|\Lambda_N|^{-1} \log \left(1 + \int F_N d\mu_{C_N} \right) \tag{205}$$

where

$$F(\Lambda_N) = I_N(\sigma_N, \Lambda_N) - 1 + K_N^N(\Lambda_N) \tag{206}$$

By (126) and (74)
$$\|I_N(\sigma_N, \Lambda_N) - 1\|_N \le \mathcal{O}(1) h^2 |\sigma_N| \le \mathcal{O}(1) h^2 r2^{-N} \tag{207}$$

and
$$\|K_N^N(\Lambda_N)\|_N \le A^{-1} \|K_N^N\|_N \le A^{-1} r2^{-N} \tag{208}$$

so that $\|F(\Lambda_N)\|_N$ is $\mathcal{O}(2^{-N})$ as $N \to \infty$.

In a following lemma we prove that for h sufficiently large $\int G_N(\Lambda_N, 0, \zeta) d\mu_{C_N}(\zeta) \le 2$. Then we estimate

$$\left| \int F_N(\Lambda_N) d\mu_{C_N} \right| \le \|F(\Lambda_N)\|_N \int G_N(\Lambda_N, 0, \zeta) d\mu_{C_N}(\zeta) \le 2 \|F(\Lambda_N)\|_N = \mathcal{O}(2^{-N}) \tag{209}$$

Hence the expression (205) is $\mathcal{O}(2^{-N}) |\Lambda_N|^{-1}$ and goes to zero very quickly as $N \to \infty$

Now we consider the first term in (204). If we replace $E_j^N(B)$ by the invariant quantity $E_j(B)$ we have

$$|\Lambda_N|^{-1} \sum_{j=1}^{N} \sum_{B \in \mathcal{B}_j(\Lambda_N)} E_j(B) = L^{-dN} \sum_{j=1}^{N} L^{d(N-j)} E_j(B) = \sum_{j=1}^{N} L^{-dj} E_j(B) \tag{210}$$

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Since $|E_j(B)| = \mathcal{O}(2^{-j})$ this converges to the infinite sum as $N \to \infty$.

Now we are left with

$$|\Lambda_N|^{-1} \sum_{j=1}^{N} \sum_{B \in \mathcal{B}_j(\Lambda_N)} (E_j^N(B) - E_j(B)) \tag{211}$$

Since $E_j^N(B) - E_j(B)$ vanishes away from the boundary the term is bounded by a constant times

$$|\Lambda_N|^{-1} \sum_{j=1}^{N} \sum_{B \in \mathcal{B}_j(\partial \Lambda_N)} 2^{-j} \leq \mathcal{O}(1) L^{-dN} \sum_{j=1}^{N} L^{(d-1)(N-j)} 2^{-j}$$

$$\leq \mathcal{O}(1) L^{-N} \sum_{j=1}^{N} L^{-(d-1)j} 2^{-j} = \mathcal{O}(L^{-N}) \tag{212}$$

where $\mathcal{B}_j(\partial \Lambda_N)$ are the boundary blocks in $\mathcal{B}_j(\Lambda_N)$. Hence this goes to zero as $N \to \infty$ to complete the proof, except for the next lemma.

Lemma 14 *For h sufficiently large*

$$\int G_N(\Lambda_N, 0, \zeta) d\mu_{C_N}(\zeta) \leq 2 \tag{213}$$

Remark. The proof is similar to the bound on $\int G_j(X, 0, \zeta) d\mu_{\Gamma_j}$ given in [2], (6.53). However

$$C_N(x - y) = \sum_{j=N+1}^{\infty} \Gamma_j(x - y) \tag{214}$$

has infinite range and so we must approach things a little differently. Note that C_N does satisfy essentially the same bound as Γ_{N+1} namely

$$|\partial^\alpha C_N(x)| \leq 2c_\alpha L^{-(d-2+|\alpha|)N} \tag{215}$$

Proof. As noted in [2], lemma 6.31, after a Sobolev inequality and a Holder inequality it suffices to show that for fixed a and any multi-index α that

$$\int \exp\left(ah^{-2} L^{(2|\alpha|-2)N} \sum_{x \in \Lambda_N^*} |(\partial^\alpha \zeta)(x)|^2 \right) d\mu_{C_N}(\zeta) \leq \exp(\mathcal{O}(h^{-2}) \tag{216}$$

With

$$A = 2ah^{-2} L^{(2|\alpha|-2)N} \; C_N^{1/2} (\partial^\alpha)^* 1_{\Lambda_N^*} \partial^\alpha C_N^{1/2} \tag{217}$$

The integral is computed as

$$\int \exp\left(\frac{1}{2} (\zeta, A\zeta) \right) d\mu_I(\zeta) = \det(1 + A)^{-1/2} \tag{218}$$

provided A is trace class. But by (215) and $|\Lambda_N^*| \leq \mathcal{O}(1)|\Lambda_N|$ we have for some constant k

$$\begin{aligned}
\mathrm{tr}\,(A) &= 2ah^{-2} L^{(2|\alpha|-2)N} \;\; \mathrm{tr}\,(1_{\Lambda_N^*} \partial^\alpha C_N (\partial^\alpha)^*) \\
&= 2ah^{-2} L^{(2|\alpha|-2)N} \sum_{x \subset \Lambda_N^*} (-1)^{|\alpha|} (\partial^{2\alpha} C_N)(0) \\
&\leq 4ah^{-2} c_{2\alpha} \sum_{x \subset \Lambda_N^*} L^{-dN} \leq kh^{-2}
\end{aligned} \tag{219}$$

Then also $\|A\| \le kh^{-2}$ and so $\operatorname{tr}(A^n) \le \|A\|_1 \|A\|^{n-1} \le k^n h^{-2n}$. Now as in (21) we have

$$\det(1+A)^{-1/2} = \exp\left(\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\operatorname{tr}(A^n)\right) \le \exp\left(\sum_{n=1}^{\infty}k^n h^{-2n}\right) \le \exp(\mathcal{O}(h^{-2})) \qquad (220)$$

This completes the proof.

A Degenerate Gaussian measures

In the text we use degenerate Gaussian measures. Here we give a precise definition.

Let Γ be a bounded symmetric operator on real-valued $\ell^2(\mathbb{Z}^d)$ that is positive in the sense that

$$(f, \Gamma f) = \sum_{x,y\in\mathbb{Z}^d} f(x)\Gamma(x,y)f(y) \ge 0 \qquad (221)$$

but only semi-definite because we allow the possibility that $(f, \Gamma f) = 0$ for some $f \ne 0$.

We want to consider a Gaussian process with covariance Γ. Since it is only semi-definite this is not quite standard. A convenient way to proceed is to let $Z(x)$ be a Gaussian process indexed by $x \in \mathbb{Z}^d$ with identity covariance, i.e. $Z(x)$ are independent normal random variables. Let $(\mathcal{M}, \mu)$ be the underlying measure space. Let $\Gamma^{1/2}(x,y) = (\delta_x, \Gamma^{1/2}\delta_y)$ be the kernel of $\Gamma^{1/2}$ and define $\phi = \Gamma^{1/2}Z$ by

$$\phi(x) = \sum_y \Gamma^{1/2}(x,y)Z(y) \qquad (222)$$

This sum converges in the $L^2(\mathcal{M}, \mu)$ since

$$\sum_y |\Gamma^{1/2}(x,y)|^2 = \Gamma(x,x) < \infty \qquad (223)$$

Expectations are integrals $\int[\cdots]d\mu$ and we use the notation

$$\int F(\phi)\, d\mu_\Gamma(\phi) \equiv \int F(\Gamma^{1/2}Z)\, d\mu(Z) \qquad (224)$$

when the integral exists. In particular if $\phi(f) = \sum_x \phi(x)f(x)$ with $f \in \ell^2(\mathbb{Z}^d)$ we have the characteristic function

$$\int \exp(i\phi(f))\, d\mu_\Gamma(\phi) = \int \exp(iZ(\Gamma^{1/2}f))d\mu(Z)$$
$$= \exp\left(-\frac{1}{2}\|\Gamma^{1/2}f\|^2\right) \qquad (225)$$
$$= \exp\left(-\frac{1}{2}(f,\Gamma f)\right)$$

which verifies that ϕ is a Gaussian process with covariance Γ.

If $\phi_1 = \Gamma_1^{1/2}Z_1$ is Gaussian with covariance Γ_1 on $(\mathcal{M}_1, \mu_1)$ and $\phi_2 = \Gamma_2^{1/2}Z_2$ is Gaussian with covariance Γ_2 on $(\mathcal{M}_2, \mu_2)$, then $\phi_1 + \phi_2$ on the product space $(\mathcal{M}_1 \times \mathcal{M}_2, \mu_1 \times \mu_2)$ gives a realization of a Gaussian process with covariance $\Gamma = \Gamma_1 + \Gamma_2$. This works because the characteristic function is

$$\int \exp\left(i(\phi_1(f) + \phi_2(f))\right)d\mu_{\Gamma_1}(\phi_1)d\mu_{\Gamma_2}(\phi_2)$$
$$= \exp\left(-\frac{1}{2}(f,\Gamma_1 f)\right)\exp\left(-\frac{1}{2}(f,\Gamma_2 f)\right) \qquad (226)$$
$$= \exp\left(-\frac{1}{2}(f,\Gamma f)\right)$$

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